Norm conditions for convergence of infinite products

L. Elsner
Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld
Germany

S. Friedland
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60607
USA

§0. Introduction

A set $\Sigma = \{A_i : i \in \mathcal{I}\}$ of $n \times n$ complex valued matrices is called an **LCP** *set* if any product $A_{i_k} \cdots A_{i_1}, i_1, ..., i_k \in \mathcal{I}$ converges as $k \to \infty$. Such sets arise in constructing self-similar objects, various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. See for example [**D-L**] and the references therein. Let $\mathcal{M}(\Sigma)$ denote the set of all finite products formed by matrices in Σ plus the identity matrix. A basic result due to G. Schechtman states that an **LCP** set is product bounded (see Theorem I of [**B-W**]). That is, if Σ is an **LCP** set then $\mathcal{M}(\Sigma)$ is a bounded set. It is well known that $\mathcal{M}(\Sigma)$ is bounded iff there exist a norm $||\cdot|| : \mathbf{C}^n \to \mathbf{R}_+$ such that all matrices in Σ are contractions with respect to this norm. Using this result and Theorem 5.1 in the paper of Daubechies-Lagarias (rephrased for **LCP**) one has a set of necessary and sufficient conditions for **LCP** sets. However this theorem is given in terms of an infinite number of conditions. The object of this paper is to find necessary and sufficient conditions for an **LCP** set in terms of the existence of a distinguished norm on $\mathbf{C}^n$. A sufficient condition of this type for a finite set Σ was given in [**E-K-N**]. It was proved that a paracontracting finite set Σ is **LCP**. Here Σ is called paracontracting (with respect to a norm $||\cdot||$) if

$$Ax \neq x \Rightarrow ||Ax|| < ||x||, x \in \mathbf{C}^n, A \in \Sigma.$$

Discussing necessary conditions, in [**Els**] it was shown that if a finite set Σ is an **LCP** set with continuous limit function then Σ is paracontracting with respect to some norm. Consult with [**D-L**] for the natural definition of a continuous limit function. Note also Theorem 4.2 of [**D-L**] which characterizes finite **LCP** sets with a continuous limit function. In [**Els**] it is conjectured that any finite **LCP** set is paracontracting. In this paper we give a necessary and sufficient norm condition on a set $\Sigma = \{A_1, A_2\}$ to be an **LCP** set.

Main Theorem. *The following conditions are equivalent.*
(i) The set $\Sigma = \{A_1, A_2\} \subset M_n(\mathbf{C})$ is an **LCP** set.
(ii) There exists a norm $||\cdot||$ on $\mathbf{C}^n$ so that
(a) $||A_i|| \leq 1, i = 1, 2;$
(b) $spec(A_i) \cap S^1 \subset \{1\}$, i=1,2;
(c) $||A_1 A_2 x|| = ||x|| \iff A_1 x = A_2 x = x.$

Here S^1 stands for the unit circle in the complex plane and $spec(A)$ denotes the spectrum of a matrix A.

§1. Preliminary results

As usual, denote by $M_{mn}(\mathbf{C})$ the vector space of all $m \times n$ matrices with complex valued entries. We let $M_n(\mathbf{C}) = M_{nn}(\mathbf{C})$. Let $\Sigma = \{A_1, ..., A_m\} \subset M_n(\mathbf{C})$. For each $B \in \mathcal{M}(\Sigma)$ we denote by $\Sigma(B) \subset \Sigma$ the subset of all matrices in Σ which appear in the product B. (If $B = I$ and $I \notin \Sigma$ then $\Sigma(B) = \emptyset$.) We wish to determine if Σ is an **LCP** set. Clearly any nontrivial strict subset $\Sigma' \subset \Sigma$ has to be an **LCP** set. Note that $card(\Sigma') < m$. Suppose we already verified that each strict subset $\Sigma' \subset \Sigma$ is an **LCP** set. Thus, to show that Σ is an **LCP** set it is left to show that any left infinite product containing each $A_i, i = 1, ..., m$, an infinite number of times converges. Set

$$\mathcal{B}(\Sigma) = \{B : B \in \mathcal{M}(\Sigma), \Sigma(B) = \Sigma\}.$$

The following lemma follows straightforward from [**B-W**].

Lemma. *Let $\Sigma \subset M_n(\mathbf{C})$ be a finite set. Assume that $\mathcal{B}(\Sigma)$ is bounded. Then any left infinite product containing each element of Σ an infinite number of times converges iff $\mathcal{B}(\Sigma)$ is an* **LCP** *set.*

Theorem 1. *Let $\Sigma = \{A_1, ..., A_m\} \subset M_n(\mathbf{C})$ be an* **LCP** *set. Assume*

$$\cap_{i=1}^m Ker(I - A_i) = \{0\}. \tag{1}$$

Then there exists a norm $||\cdot||$ on $\mathbf{C}^n$ and $0 < q < 1$ such that

$$||A_i|| \leq 1, i = 1, ..., m, \tag{2}$$

$$||B|| \leq q, B \in \mathcal{B}(\Sigma). \tag{3}$$

In particular, any infinite product of matrices in Σ containing each A_i an infinite number of times converges to zero.

Proof. Clearly, $\mathcal{B}(\Sigma)$ is an **LCP** set. Consider some left infinite product from the elements of $\mathcal{B}(\Sigma)$ which converges to a matrix denoted by C. As each A_i appears in this product