Resultants and the Continuity of the Join Pairing on Chow Varieties

Judith Plümer[1]

Introduction

In [13] Lawson showed the equivalence of the stable Chow variety $\mathcal{C}$ and the product $\prod_{i>0} K(\mathbb{Z}, 2i) =: K(\mathbb{Z}, even)$ of Eilenberg MacLane spaces $K(\mathbb{Z}, 2i)$. The invested complex suspension theorem is fundamental for the definition of Lawson homology [7]. The original proof of this theorem uses geometric measure theory. Here an algebraic and as we think more elementary proof (Thm. 3.1) is given going back to the construction of Chow varieties via resultants [5]. Once the basic continuity statements are proved, we can stay close to the geometric intuition [13], which is behind the measure theoretic arguments, by the Chow/Van der Waerden approach to Chow varieties. The observation is that one can calculate the Chow form of a suspended cycle (Prop. 2.2) from the original Chow form. Thus suspension is an algebraic map.

To prove that the join pairing is algebraic (Thm. 2.3), hence continuous, one just has to combine the continuity of the complex suspension with classical results (Barlet [1]). (Independently Barlet [2] obtained a proof of Thm. 2.3 using different methods.)

In [4] the join pairing was used to define an infinite loop space structure on $K(\mathbb{Z}, even)$ which is compatible with the infinite loop space structure on BU induced by Whitney sum. The required continuity checks (Lemma 4.5) for the operation of [4] also follow easily in the Chow/Van der Waerden picture.

Section 1 describes Chow forms in the terms of Chow and Van der Waerden and some properties of them. In section 2 the complex suspension and join of cycles and several stabilizations are defined. The Chow form of a suspended cycle is determined from a given Chow form and continuity of the suspension and the join pairing are shown. Section 3 contains a proof of the complex suspension theorem. Finally the last section recollects the continuity statements which make the total Chern class map an infinite loop map.

Throughout this paper we work in the category kTop of compactly generated, weak Hausdorff spaces [17]. All proofs are given for algebraic cycles with support in $\mathbb{CP}^n$ but they directly work for cycles with support in an algebraic subset $X \subset \mathbb{CP}^n$ (see also [13]), except that one has to take care of π_0–phenomena (compare [13, p. 265]).

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1 Review of the Chow – Van der Waerden approach to Chow varieties

There are many different ways to define the Chow form of a variety $V \subset \mathbb{CP}^n$ ([5],[11],[8]). The definition of Chow and Van der Waerden is constructive and therefore very useful for proofs of continuity. First some notations and basic definitions:

1.1 Notation: Let $\mathbb{C} \subset K \subset \Omega$ be fields.

i. A projective algebraic set $V_K \subset \Omega\mathbb{P}^n$ *over* K is the set of common zeros of a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$.
ii. Let $V_K \subset \Omega\mathbb{P}^n$ be a projective algebraic set. The homogeneous ideal $I_K(V)$ which is generated by

$$\{f \in K\,[x_0, \ldots, x_n] \mid f \text{ homogeneous and } f(\underline{v}) = 0\ \forall \underline{v} \in V\}$$

in $K\,[x_0, \ldots, x_n]$ is the *vanishing ideal* of V_K.

An algebraic set $V_K \subset \Omega\mathbb{P}^n$ is called *irreducible* or *a variety* iff $I_K(V) \subset K[x_0, \ldots, x_n]$ is prime. The dimension of a variety V_K is its dimension as a topological subspace of $\Omega\mathbb{P}^n$ in the Zariski topology. It follows from the Noetherian Normalization Theorem [12, Chap. II, Thm. 3.1] that the dimension of $V_K \subset \Omega\mathbb{P}^n$ is independent of the extension field Ω of K, that means

$$\dim V_K = \dim V_K \cap K\mathbb{P}^n\,.$$

From now on Ω will denote a fixed *universal field* in the sense of Van der Waerden [21], given as the algebraic closure of $\mathbb{C}(U)$, where U is a countably infinite set of algebraically independent elements over $\mathbb{C}$. K is always a subfield of Ω with finite transcendence degree over $\mathbb{C}$.

Suppose $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in \Omega^{n+1}$. Then one can define $K[\underline{\xi}] := K[\xi_0, \ldots, \xi_n]$. If $\underline{\xi} \in \Omega\mathbb{P}^n$ then $K[\underline{\xi}]$ depends on the choice of a representative of $\underline{\xi}$ in Ω^{n+1}. The definition of a special representative of points in $\Omega\mathbb{P}^n$ makes this ring adjunction unique up to canonical isomorphisms:

1.2 Definition: Let $(\xi_0, \ldots, \xi_n) = \underline{\xi} \in \Omega\mathbb{P}^n$ and i minimal such that $\xi_i \neq 0$. Then

$$\tau\underline{\xi} := \left(\tau\frac{\xi_0}{\xi_i}, \ldots, \tau\frac{\xi_i}{\xi_i}, \ldots, \tau\frac{\xi_n}{\xi_i}\right) \in \Omega^{n+1},\ \tau \in \Omega - \overline{K(\xi_0, \ldots, \xi_n)}.$$

For every point $\underline{\xi} \in \Omega\mathbb{P}^n$ the ring $K[\tau\underline{\xi}]$ depends on choices up to canonical isomorphisms, which extend to K–automorphisms of Ω. From now on the condition $f(\underline{\xi}) = 0,\ f \in K[x_0, \ldots, x_n]$, is understood as $f(\tau\underline{\xi}) = 0$ for $\underline{\xi} \in \Omega\mathbb{P}^n$. Also one defines $K(\tau\underline{\xi})$ to be the quotient field of $K[\tau\underline{\xi}]$.

The advantage of viewing varieties in $\mathbb{CP}^n$ as varieties in $\Omega\mathbb{P}^n$ *over* $\mathbb{C}$ is, that varieties in $\Omega\mathbb{P}^n$ over K are completely described by single points in $\Omega\mathbb{P}^n$.