GEOMETRY OF 1-TORI IN GL_n

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Introduction

Geometry of unipotent (long) root subgroups in Chevalley groups and related geometries have been extensively studied and are very well understood, see, for example, [AS], [Co1] – [Co4], [Cu], [Ka1], [LS], [Mc], [Ne], [Ti1] – [Ti4], [W1], [ZS1] and references in [Ka2], [Kn], [Z1]. At the same time very little is known about the geometry of tori. Anisotropic groups do

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not have unipotents and in this case the study of tori is a necessity. Here one could mention, for example, very interesting results of B.N.Weisfeiler and D.G.James [We1] – [We3], [JW]. However even for Chevalley groups it might be interesting to describe orbits on pairs, triples, etc. of tori of various classes, identify corresponding spans, classify irreducible subgroups generated by tori from this class, etc. This may lead to geometries similar to, but different from buildings and their residues (compare [We3], p.365). Results in this spirit might shed a new light on the known results concerning description of subgroups (see the surveys [Kn], [V7] – [V9], [Z1] – [Z3]). They would be also the first step towards generalization of results on subgroups generated by pseudoreflections, two-dimensional elements or the like [C1], [C2], [Cox], [Hu], [HW1] – [HW3], [Ko1] – [Ko3], [ST], [W2], [W3], [Wa], [ZS2], [ZS3] to other types of groups and to other types of semisimple elements.

In this paper we study in some detail the model case of *1-tori* in the general linear group $G = \mathrm{GL}(n, T)$ over a skew-field T. The 1-tori are one parameter groups of pseudoreflections with the same axis and centre. First we study the orbits of G acting by conjugation on pairs (X, Y) of 1-tori (referred to as *orbitals* in the sequel). It turns out, that when the skew-field T contains at least 7 elements, there is a close analogy with the root subgroups. Namely, as for the root subgroups one can define the angle ϕ between two 1-tori X and Y, taking values 0, $\pi/3$, $\pi/2$, $2\pi/3$ and π. When $\phi < \pi$ the above orbitals are almost completely defined by the angle between X and Y. (As for the root subgroups the angle $\pi/3$ leads to two possible configurations, fused by the external automorphism of G.) Unlike root subgroups the angle π leads to $|T| - 1$ distinct orbitals, falling into two essentially different cases, in one of which the span $\langle X, Y\rangle$ is isomorphic to $\mathrm{GL}(2, T)$, in another one it is triangularizable. There is another important aspect in which this result differs from the case of root subgroups. Namely, whereas for pairs (X, Y) of root subgroups the orbitals and the corresponding spans $\langle X, Y\rangle$ do not depend on the skew-field, in our case the fields K with $|K| \leq 7$ are true exceptions. In these cases the shape of the span $\langle X, Y\rangle$ for a given orbital differs from the generic case.

We show how the above result immediately implies description of the irreducible subgroups in G generated by 1-tori (the analogue of McLaughlin's theorem for pseudoreflections, see [V6]) and description of overgroups of the diagonal subgroup $D = \mathrm{D}(n, T)$ (see [B], [BV], [V1] for the first proof of this result; [Se] for a proof for finite fields; [V2], [V4] for different proofs; [BK], [K1] – [K4], for the description in the exceptional cases $|K| \leq 5$; [Fr], [L1] for proofs of a weeker statement about the maximality of the monomial subgroup; [V3], [Ki] for the case of the special linear group; and surveys [Z1] – [Z3], [V7] – [V9] for the general background and further references).