Hyperbolic 3-manifolds as 2-fold coverings according to Montesinos

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Abstract

The Fibonacci manifolds and the ten smallest known compact orientable three-dimensional hyperbolic manifolds are obtained as two-fold coverings of the three-dimensional sphere. The corresponding branch sets are described. It is shown that every manifold under consideration admits a Heegaard splitting of genus two.

In this paper we consider compact orientable hyperbolic three-dimensional manifolds. We are interesting in two families of such manifolds. First of them are the Fibonacci manifolds M_n, $n \geq 2$, uniformized by the Fibonacci groups $F(2,2n)$. Manifolds M_n were considered by J. Mennicke, H. Helling and A. C. Kim in [20] and [8]. These manifolds are hyperbolic for $n \geq 4$. It was shown in [11] that M_n can be obtained as the n-fold cyclic covering of the 3-sphere, branched over the figure-eight knot. In the previous preprint [19] we showed that for any $n \geq 2$ the volume of the compact hyperbolic Fibonacci manifold M_{2n} is exactly equal to the volume of the non-compact manifold $S^3 \setminus Th_n$, where the Turk's head link Th_n is a closed 3-strings braid $(\sigma_1 \sigma_2^{-1})^n$. In section 1 we will prove that the Fibonacci manifold M_n, $n \geq 2$, can be obtained as a two-fold covering of the 3-sphere, branched over the Turk's head link Th_n.

In section 2 we will discuss the second family of compact 3-manifolds $W(m,n;p,q)$ which can be obtained by (m,n) and (p,q) Dehn surgeries on two components of the Whitehead link. According to the Montesinos theorem [22], any manifold, obtained by Dehn surgeries on a strongly invertible link can be presented as a 2-fold covering of the 3-sphere, branched over some link.

Moreover there is an algorithm for constructing this branch set in [22]. We will apply this algorithm for describing manifolds $W(m,n;p,q)$ as a 2-fold branched coverings of 3-sphere. In particular, by this way we will get the ten smallest known compact orientable hyperbolic 3-manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{10}$ from [13].

Section 3 is devoted to the conjecture of R. Meyerhoff and W. Neumann [21]. Comparing results of section 1 and section 2 we will prove that the third smallest manifold $\mathcal{M}_3$, constructed in [21], is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$ and its volume is exactly equals to the volume of the regular ideal tetrahedron in Lobachevsky space.

In section 4 we will discuss the Heegaard genus of above manifolds. More exactly, we will prove that the Heegaard genus of the Fibonacci hyperbolic manifolds and of the ten smallest known hyperbolic manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{10}$ from [13] is equal two.

1. Fibonacci manifolds as two–fold coverings

The Fibonacci groups

$$F(2,m) \; = \; \langle x_1, \, x_2, \, \ldots, \, x_m \; \mid \; x_i \, x_{i+1} \; = \; x_{i+2}, \; i \bmod m \rangle.$$

were introduced by J. Conway [6]. It was shown by H. Helling, A. C. Kim and J. Mennicke [8] that the groups $F(2,2n)$, $n \geq 2$, are interesting from the geometrical point of view. The group $F(2,2n)$ is isomorphic to a properly discontinuous cocompact group of isometries which acts without fixed points on a space X_n, where $X_2 = S^3$ – spherical, $X_3 = \mathbb{E}^3$ – Euclidean and $X_n = \mathbb{H}^3$ – hyperbolic for $n \geq 4$.

We will call a three-dimensional manifold $M_n = F(2,2n)$, $n \geq 2$, uniformized by the Fibonacci group to be a *Fibonacci manifold.*

Various properties of hyperbolic Fibonacci manifolds M_n, $n \geq 4$ were investigated in [8], [10], [11] and [18].

For describing geometrical properties of the Fibonacci manifolds we need to define one family of knots and links. Denote by Th_n, $n \geq 2$, the closed 3-strings braid $(\sigma_1 \sigma_2^{-1})^n$, where σ_1 and σ_2 are canonical generators [2]. We note that Th_n is a 3-component link if n is divided by three and it is a knot otherwise. This family contains well-known knots and links. In particular, Th_2 is the figure-eight knot, Th_3 are the Borromean rings, Th_4 is the Turk's head knot 8_{18} and Th_5 is the knot 10_{123}.

It was shown by W. Thurston [27] that a complement $S^3 \setminus Th_n$, $n \geq 2$, is a hyperbolic manifold. In [16] the virtual Betti numbers of compact hyperbolic manifolds constructed by Dehn surgeries on knots Th_n were studied. The symmetry groups of knots and links Th_n are described in [25]. In [19] authors proved that the volume of $S^3 \setminus Th_n$ is exactly equal to the volume of the Fibonacci manifold M_{2n}.