The Combinatorial Structure of Small Cut and Metric Polytopes

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Abstract. We study the combinatorial structure of the cut and metric polytopes on n nodes for $n \leq 5$. Those two polytopes have a complicated geometrical structure, but using their large symmetry group, we can completely describe their face lattices. We present, for any n, some orbits of faces and give new result on the tightness of the wrapping of the cut polytope by the metric polytope, disproving a conjecture of [14] on their lattices.

Key words: metric polytope, complete bipartite subgraphs polytope, face lattice.

1 Introduction

We first recall the definition of the *metric polytope* m_n, the *cut polytope* c_n, and their relatives, the *metric cone* and the *cut cone*. Then we present some applications to well known optimization problems and some combinatorial and geometric properties of those polyhedra. The general references are [4, 24] for polytopes and [5] for graphs. For a complete study of the applications and the combinatorial optimization aspects of those polyhedra, we refer, respectively, to the surveys [13] and [21].

For all 3-sets $\{i,j,k\} \subset \{1,\ldots,n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \qquad (1)$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \qquad (2)$$

The inequalities (1) define the metric cone and the metric polytope m_n is obtained by bounding the latter by the inequalities (2). The $3\binom{n}{3}$ facets defined by the inequalities (1),

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which can be seen as triangle inequalities for distance x_{ij} on $\{1,2,\ldots,n\}$, are called *homogeneous triangle facets* and are denoted by $Tr_{ij,k}$. The $\binom{n}{3}$ facets defined by the inequalities (2) are called *non-homogeneous triangle facets* and are denoted by Tr_{ijk}.

Given a subset S of $V_n = \{1,2,\ldots,n\}$, the *cut* determined by S consists of the pairs (i,j) of elements of V_n such that exactly one of i, j is in S. $\delta(S)$ denotes both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$, that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of language, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope of the complete graph c_n, which is also called the *complete bipartite subgraphs polytope*, is the convex hull of all 2^{n-1} cuts, and the cut cone is the conic hull of all $2^{n-1}-1$ nonzero cuts. Those polyhedra were considered by many authors, see for instance [2, 3, 9, 11, 12, 13, 14, 15, 17, 18] and references therein. One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the max-cut and multicommodity flow problems.

Given a graph $G = (V_n, E)$ and nonnegative weights w_e, $e \in E$, assigned to its edges, the *max-cut* problem consists in finding a cut $\delta(S)$ whose weight $\sum_{e \in \delta(S)} w_e$ is as large as possible. It is a well-known NP-complete problem. By setting $w_e = 0$ if e is not an edge of G, we can consider without loss of generality the complete graph on V_n. Then the max-cut problem can be stated as a linear programming problem over the cut polytope c_n as follows:

$$\begin{cases} \max \quad w^T \cdot x \\ x \in c_n. \end{cases}$$

Since the metric polytope is a relaxation of the cut polytope, optimizing $w^T \cdot x$ over c_n instead of m_n provides an upper bound for the max-cut problem [3].

With E the set of edges of the complete graph on V_n, an instance of the *multicommodity flow problem* is given by two nonnegative vectors indexed by E: a capacity $c(e)$ and a requirement $r(e)$ for each $e \in E$. Let $U = \{e \in E : r(e) > 0\}$. If T denotes the subset of V_n spanned by the edges in U, then we say that the graph $G = (T, U)$ denotes the *support* of r. For each edge $e = (s,t)$ in the support of r, we seek a flow of $r(e)$ units between s and t in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a flow exists, we call c, r *feasible*. A necessary and sufficient condition for feasibility is given by the Japanese theorem [16]: a pair c, r is feasible if and only if $(c-r)^T x \geq 0$ is valid over the metric cone. For example, $Tr_{ij,k}$ can be seen as an elementary solvable flow problem with $c(ij) = r(ik) = r(jk) = 1$ and $c(e) = r(e) = 0$ otherwise, so the inequalities (1) correspond to $(c-r)^T x \geq 0$ for x in the metric cone. Therefore, the metric cone is the dual cone to the cone of feasible multicommodity flow problems.