The numerical computation of homoclinic orbits for maps

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Abstract. Transversal homoclinic orbits of maps are known to generate shift dynamics on a set with Cantor like structure. In this paper a numerical method is developed for computation of the corresponding homoclinic orbits. They are approximated by finite orbit segments subject to asymptotic boundary conditions. We provide a detailed error analysis including a shadowing type result by which one can infer the existence of a transversal homoclinic orbit from a finite segment. This approach is applied to several examples. In some of them parameters appear and closed loops of homoclinic orbits are found by a path-following algorithm.

Key words. Dynamical systems, homoclinic points for maps, numerical methods, shadowing.

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1 Introduction

One of the fundamental results on chaotic behavior in discrete dynamical systems is Smale's Homoclinic Theorem, see [23], [21], [22]. For a more recent overview of homoclinic orbits, their bifurcations and the history of their discovery we refer to [19].

Consider a (time-)discrete system

$$x_{n+1} = f(x_n), \quad n \in \mathbb{Z} \tag{1.1}$$

with a C^l-diffeomorphism $f : \mathbb{R}^k \to \mathbb{R}^k$ and assume that $\xi \in \mathbb{R}^k$ is a hyperbolic fixed point of f, i.e. the Jacobian $f'(\xi)$ has no eigenvalues on the unit circle. Further assume that x_0 is a transversal homoclinic point, which means that the orbit generated by (1.1) satisfies

$$\lim_{n\to+\infty} x_n = \lim_{n\to-\infty} x_n = \xi \tag{1.2}$$

and the stable and unstable manifolds of ξ intersect transversally at x_0. Then the theorem states that there exists a compact set M and an integer p such that the p-th iterate of f, denoted by f^p, leaves M invariant and is topologically conjugate on M to the Bernoulli shift on two symbols. It is remarkable that this chaotic behavior on a certain subset is created by a homoclinic point that is transversal, a property that persists under the perturbation of system (1.1).

The perturbation stability of transversal homoclinic points suggests that we should be able to compute these points numerically in a robust and stable way. This is the topic of the

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present paper. Instead of homoclinic points our approach aims at computing the complete homoclinic orbit $x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}}$ by solving the 'boundary value problem' (1.1), (1.2). Any direct method for the single homoclinic point x_0 implicitly tries to solve this boundary value problem and hence is prone to the usual difficulties of shooting-type methods which are caused by exponential divergence of trajectories.

Therefore, we propose to approximate the infinite orbit by a finite orbit segment $x_J = (x_{n_-}, \ldots, x_{n_+})$, $J = \{n_-, \ldots, n_+\}$ which satisfies a 'finite boundary value problem'

$$x_{n+1} = f(x_n), \quad n = n_-, \ldots, n_+ - 1, \tag{1.3}$$

$$b(x_{n_-}, x_{n_+}) = 0. \tag{1.4}$$

Here (1.4) is a general set of boundary conditions defined by a smooth mapping $b : \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$. Together, (1.3) and (1.4) comprise a set of $(n_+ - n_- + 1)k$ nonlinear equations for the same number of unknowns. We solve this system by Newton's method and take advantage of the sparsity pattern of the corresponding Jacobian.

The most important examples for (1.4) are periodic boundary conditions

$$b(x_{n_-}, x_{n_+}) = x_{n_+} - x_{n_-} = 0 \tag{1.5}$$

and projection boundary conditions

$$b(x_{n_-}, x_{n_+}) = (b_-(x_{n_-}), b_+(x_{n_-})) = 0 \tag{1.6}$$

where the zero sets of b_- and b_+ are linear approximations to the local unstable and stable manifold of the fixed point.

The whole approach closely mimics the well known methods for approximating homoclinic orbits of parametrized continuous systems $\dot{x} = f(x, \lambda)$ on finite intervals [9]. [4], [2], [11]. But there is one important difference to keep in mind. In the discrete case, stable and unstable manifolds can, and in the generic sense do, intersect transversally at a homoclinic point while in the continuous case both tangent spaces always contain the flow direction. If the manifolds intersect at all in a continuous system they do so in a whole curve obtained by shifting the phase of the orbit. Therefore, one parameter is needed for homoclinic orbits to occur generically in a continuous system and an additional constraint is needed to fix the arbitrary phase.

The paper is organized as follows. In section 2 we collect the necessary prerequisites from the theory of exponential dichotomies for difference equations. This section is largely based on the paper of Palmer [21] where exponential dichotomies are used to prove the Shadowing Lemma and Smale's Homoclinic Theorem. Earlier references on dichotomies for difference equations include [5] and [15].

Section 3 contains the basic results of this paper. In a neighborhood of a transversal homoclinic orbit we find a unique solution of the equations (1.3), (1.4) provided b satisfies a nondegeneracy condition and n_-, n_+ are taken sufficiently large. We also show error estimates which are analogous to the continuous case.

In section 4 we treat several examples including the Poincaré map for the Duffing system. We also compute branches of discrete homoclinic orbits in cases where the system (1.1)