Reflective integral lattices

Rudolf Scharlau Britta Blaschke

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Abstract

A lattice L with a positive definite quadratic form is called reflective if the unique largest subgroup generated by reflections of the orthogonal group $\mathrm{O}(L)$ has no fixed vector. Equivalently, the "root system" $\mathsf{R}(L)$ has maximal rank. The root system of a lattice is defined in Section 1 below; the roots are not necessarily of length 1 or 2. In Section 2 the structure of reflective lattices is worked out. They are described and classified by pairs $(\mathsf{R}, \mathcal{L})$, where R is a "scaled root system", and the "code" $\mathcal{L}$ is a subgroup of the "reduced discriminant group" $\overline{T}(\mathsf{R})$. The crucial point is that $\overline{T}(\mathsf{R})$ only depends on the combinatorial equivalence class of the root system R. In Section 3 we give a precise description of the full root system of a reflective lattice if one starts with a sub root system of combinatorial type $n\mathsf{A}_1$ or $m\mathsf{A}_2$. In Section 4 our techniques are applied to a complete and explicit description of all reflective lattices in dimensions ≤ 6.

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1 Introduction

We consider positive definite, integral quadratic lattices L. That is, L is a free $\mathbf{Z}$-module of finite rank, $L \cong \mathbf{Z}^n$, together with a positive definite symmetric bilinear form. The value of the form at vectors $x, y \in V := \mathbf{Q}L$ is denoted by $(x, y) \in \mathbf{Q}$. "Integral" means that $(x, y) \in \mathbf{Z}$ for all $x, y \in L$.

For $v \in V$, the reflection

$$s_v : x \longmapsto x - \frac{2(v,x)}{(v,v)} v$$

is an isometry of the quadratic vector space V.

Recall that a vector $v \in L$ is called *primitive* if $v/m \notin L$ for all integers $m > 1$.

Definition 1.1 A vector $v \in L$ is called a *root* of L if it is primitive and if s_v maps L into itself. A lattice is called *reflective* if its roots generate a sublattice of full rank.

The following easy observation is the starting point of this paper.

Proposition 1.2 *The set of all roots of L*

$$\mathsf{R}(L) := \{ v \in L \mid v \textit{ primitive} \ , \ s_v \in \mathrm{O}(L) \}$$

is a root system in the usual sense of Lie algebra theory.

Proof. We refer to [Bou68, Chap. VI] for the definition of a root system. If $v, v' \in \mathsf{R}(L)$, then $s_v v' \in \mathsf{R}(L)$ since $s_{s_v v'} = s_v s_{v'} s_v$. By the above reflection formula, a primitive vector $v \in L$ is a root if and only if $(v, L) \subseteq \mathbf{Z}(v, v)/2$. In particular, the "crystallographic condition", $2(v, v')/(v, v) \in \mathbf{Z}$ for any two $v, v' \in \mathsf{R}(L)$, holds. $\square$

We shall consider $\mathsf{R}(L)$ as a root system R together with a specified quadratic form invariant under the Weyl group $\mathrm{W}(\mathsf{R}) = \langle s_v \mid v \in \mathsf{R} \rangle$, and the notion of isomorphism of root systems is that of an isometric bijection. Thus, $\mathsf{R}(L)$ is even a finer invariant than an ordinary root system; we shall sometimes speak of a *scaled root system.*

We observe that v is a root if and only if v is an element of $(v, v)/2$ times the *dual lattice* $L^\# = \{ y \in V \mid (L, y) \subseteq \mathbf{Z} \}$. In particular, $(v, v)/2$ divides the exponent of the finite group $T(L) := L^\#/L$, the so-called *discriminant group* of L. If L is self-dual (unimodular), then the roots are precisely the vectors of norm $(v, v) = 1$ or 2. In general, all divisors of twice the exponent of $T(L)$ may occur as norms of roots (think of lattices with an orthogonal basis).

As examples, consider the two quadratic forms given by the following matrices (the indices at the vectors indicate the value (v, v)):