The spectrum of a locally coherent category

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Abstract: A topology on the spectrum of a locally coherent Grothendieck category is introduced. The closed subsets are related to certain localizing subcategories which are characterized in terms of Serre subcategories of the full subcategory of finitely presented objects.

A Grothendieck category $\mathcal{A}$ is said to be locally coherent provided that $\mathcal{A}$ has a generating set of finitely presented objects and the full subcategory $\mathrm{fp}(\mathcal{A})$ of finitely presented objects in $\mathcal{A}$ is abelian. The spectrum $\mathrm{sp}(\mathcal{A})$ of $\mathcal{A}$ is a representative set of indecomposable injective objects in $\mathcal{A}$. We show that this set carries a natural topology and it is the purpose of this paper to establish a natural and bijective correspondence between the following structures which arise for each locally coherent category $\mathcal{A}$:

– Serre subcategories of $\mathrm{fp}(\mathcal{A})$,

– hereditary torsion theories of finite type for $\mathcal{A}$,

– closed subsets of $\mathrm{sp}(\mathcal{A})$.

This analysis is motivated by some construction which reduces the theory of purity for a locally finitely presented category with products to the study of injectives in a locally coherent category (e.g. [2]). Model theory of modules is a classical example of this theory. For instance, given a ring Λ the category $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of additive functors from the category of finitely presented Λ^{op}-modules to the category of abelian groups is locally coherent, and the functor

$$\mathrm{Mod}(\Lambda) \to (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad M \mapsto M \otimes_\Lambda -$$

identifies the pure-injective Λ-modules with the injective objects of $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$.

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1. Locally finitely presented abelian categories

We recall some terminology and some well-known facts about locally finitely presented categories and Grothendieck categories. Let $\mathcal{A}$ be an additive category with direct limits. An object $X \in \mathcal{A}$ is *finitely presented* provided that $\mathrm{Hom}(X, \ )$ commutes with direct limits and we denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of finitely presented objects in $\mathcal{A}$. The category $\mathcal{A}$ is said to be *locally finitely presented* if the isomorphism classes of $\mathrm{fp}(\mathcal{A})$ form a set and every object in $\mathcal{A}$ is a direct limit of objects in $\mathrm{fp}(\mathcal{A})$. An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [1, Satz 1.5], [2, 2.4].

Fix now a locally finitely presented abelian category $\mathcal{A}$. An object in $\mathcal{A}$ is *finitely generated* if it is a quotient of some finitely presented object and it is easily seen that any object in $\mathcal{A}$ is a direct limit of finitely generated subobjects. In particular, one has the following lemma.

Lemma 1.1 *An object $X \in \mathcal{A}$ is finitely generated iff for any epimorphism $\varphi\colon Y \to X$ there is a finitely generated subobject U of Y such that $\varphi(U) = X$.*

Proof: Straightforward.

The category $\mathcal{A}$ is completely determined by $\mathrm{fp}(\mathcal{A})$. In fact, $\mathrm{fp}(\mathcal{A})$ is additive, has cokernels and the functor

$$\mathcal{A} \to \mathrm{Lex}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X)|_{\mathrm{fp}(\mathcal{A})}$$

from $\mathcal{A}$ into the category of additive left exact functors from $\mathrm{fp}(\mathcal{A})^{\mathrm{op}}$ to Ab is an equivalence [1, Satz 2.4]. Given a full additive subcategory $\mathcal{C}$ of $\mathrm{fp}(\mathcal{A})$ we denote by $\vec{\mathcal{C}}$ the full subcategory of $\mathcal{A}$ which consists of direct limits $\varinjlim X_i$ with $X_i \in \mathcal{C}$ for all i. Note that this subcategory is closed under direct limits in $\mathcal{A}$ and that the finitely presented objects in $\vec{\mathcal{C}}$ are precisely the direct summands of objects in $\mathcal{C}$ [2, Theorem 4.1]. Finally, we recall that $\mathcal{A}$ is *locally coherent* provided that finitely generated subobjects of finitely presented objects are finitely presented, equivalently if $\mathrm{fp}(\mathcal{A})$ is abelian [5, Proposition 2.2].