On the tameness of certain 2-point algebras

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Abstract

In [Ge1] it was proved that all tame distributive 2-point algebras are factors of algebras contained in a small list given in that paper. Also there were given tameness proofs for the algebras of that list with the exception of essentially two cases. It was pointed out already in that paper that in order to complete the tameness proofs one essentially had to study two 2-point algebras with homogenous relations, call them Λ and $\Gamma_{m,A}$, see 1.2 for details. The subject of this paper is to supply these proofs. Indeed we manage in the case of Λ to show that the universal Galois covering with group $\mathbb{Z}$ is locally support-finite and tame, thus so is Λ. In the case of $\Gamma_{m,A}$ we use the generalized 1-point extensions of [Dr3], to prove that the universal Galois covering with group $\mathbb{Z}$ is tame, and the tameness of the algebra $\Gamma_{m,A}$ itself will follow from a result announced by Drozd ([Dd]).

1 Introduction

1.1. Let us first fix some notations. k will be an algebraically closed field. In this paper by a k-algebra we mean a locally bounded spectroid in the sense of [GR]. Usually we present such a category Λ in the form $k[Q]/I$ where $k[Q]$ is the k-linear hull of the path category of a quiver Q and I is an admissable ideal. We denote by Λ-mod the category of contravariant k-linear functors $\Lambda \to k$-mod. We compose k-linear maps from right to left but morphisms in other categories as well as arrows from left to right.

The (Jacobson) radical of a locally bounded spectroid Λ is denoted by $\operatorname{Rad}\Lambda$.

1.2. Consider the quiver $Q:\ \sigma \circlearrowright a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \circlearrowleft \rho$ and the categories

$$\Lambda 9'_{m,A} := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu - \rho^m(1_b - A\rho), \sigma^2\nu, \rho^2\gamma, \sigma^4, \rho^{m+2}\rangle$$

$$\Lambda 9''_{m,A} := k[Q]/\langle \sigma\nu - A\nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu - \rho^m, \sigma^2\nu, \rho^2\gamma, \sigma^3, \rho^{m+1}\rangle$$

$$\Gamma_{m,A} := k[Q]/\langle \sigma\nu - A\nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu, \sigma^3, \rho^{m+1}\rangle$$

$$\Lambda 10 := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \gamma\nu - \rho^2, \rho\gamma\rangle$$

$$\Lambda 10'_1 := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \gamma\nu - \rho^2, \rho\gamma, \sigma^2\nu\rangle$$

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Observe that $\Lambda 9'_{m,A}$ is selfinjective, modulo its socle it is isomorphic to $\Lambda 9''_{m,A}$, and this in turn degenerates into $\Gamma_{m,A}$, thus by [Ge2] the tameness of $\Gamma_{m,A}$ will imply the tameness of $\Lambda 9''_{m,A}$ and $\Lambda 9'_{m,A}$. Similarily $\Lambda 10$ has a projective-injective module, modulo its socle it is isomorphic to $\Lambda 10'_1$; see [Ge1, 4.2, 4.3]. For brevity we write $\Lambda 10'_1 =: \Lambda$. Concerning $\Gamma_{m,A}$ we note that $m \in \mathbb{N}$ with $m \geq 3$ and $0 \neq A \in k$. We point out that $\operatorname{Rad}^3 \Lambda = 0$ and that the elements of $\operatorname{Rad}^3 \Gamma_{m,A}$ are polynomials in ρ.

Now consider the quiver $\tilde{Q}$:

$$\begin{tikzcd}
\cdots & a_{-1} \arrow[r, "\sigma_{-1}"] \arrow[dr, "\nu_{-1}"] & a_0 \arrow[r, "\sigma_0"] \arrow[dr, "\nu_0"] & a_1 \arrow[r, "\sigma_1"] \arrow[dr, "\nu_1"] & a_2 & \cdots \\
\cdots & b_{-1} \arrow[r, "\rho_{-1}"'] \arrow[ur, "\gamma_{-1}"] & b_0 \arrow[r, "\rho_0"'] \arrow[ur, "\gamma_0"] & b_1 \arrow[r, "\rho_1"'] \arrow[ur, "\gamma_1"] & b_2 & \cdots
\end{tikzcd}$$

and the categories

$$\tilde{\Lambda} := k[\tilde{Q}]/\langle \sigma_i\nu_{i+1} - \nu_i\rho_{i+1}, \nu_i\gamma_{i+1} - \sigma_i\sigma_{i+1}, \gamma_i\nu_{i+1} - \rho_i\rho_{i+1}, \rho_i\gamma_{i+1},$$

$$\sigma_i\sigma_{i+1}\nu_{i+2} \mid i \in \mathbb{Z}\rangle$$

$$\tilde{\Gamma}_{m,A} := k[\tilde{Q}]/\langle \sigma_i\nu_{i+1} - A\nu_i\rho_{i+1}, \nu_i\gamma_{i+1} - \sigma_i\sigma_{i+1}, \rho_i\gamma_{i+1} - \gamma_i\sigma_{i+1}, \gamma_i\nu_{i+1},$$

$$\sigma_i\sigma_{i+1}\sigma_{i+2}, \rho_i \cdots \rho_{i+m}\rho_{i+m+1} \mid i \in \mathbb{Z}\rangle$$

There is a natural action of the (additive) group $\mathbb{Z}$ on these categories inducing Galois coverings $\tilde{\Lambda} \to \Lambda$ and $\tilde{\Gamma}_{m,A} \to \Gamma_{m,A}$.

2 Λ is tame

2.1. We will show, that the Galois-covering $\tilde{\Lambda}$ of Λ is locally support-finite and tame, which implies by [Pe, 3.2] the tameness of Λ (the hypothesis k uncountable may be omitted).

We first have to introduce some notation. For $i \leq j$ we set

$$\tilde{\Lambda}^{(i,j)} := \tilde{\Lambda}\{a_i, b_i, a_{i+1}, \ldots, a_{j+1}, b_{j+1}\},$$

the full subcategory of $\tilde{\Lambda}$ that contains the objects $\{a_i, \ldots, b_{j+1}\}$; for brevity write $\tilde{\Lambda}^{(i,i)} =: \tilde{\Lambda}^{(i)}$,

$$\tilde{\Lambda}^{[i]} := \tilde{\Lambda}\{a_{i-1}, a_i, b_i, a_{i+1}, b_{i+1}, b_{i+2}\}.$$

We also give names to some indecomposable regular $\tilde{\Lambda}^{(i)}$-modules. For $C \in P_1(k) := k \cup \{\infty\}$ we set $M_C^{(i)}(x) := k$ for $x \in \tilde{\Lambda}^{(i)}$, and for $C \in k$ we take $M_C^{(i)}(\gamma_i) := C \cdot \mathrm{id}_k$, $M_C^{(i)}(\sigma_i) = M_C^{(i)}(\rho_i) = M_C^{(i)}(\nu_i) = \mathrm{id}_k$, while $M_\infty^{(i)}(\rho_i) = 0$, $M_\infty^{(i)}(\gamma_i) = M_\infty^{(i)}(\nu_i) = M_\infty^{(i)}(\sigma_i) = \mathrm{id}_k$; note that $M_0^{(i)}$ and $M_\infty^{(i)}$ lie in different inhomogenous tubes $\mathcal{R}_0^{(i)}$ and $\mathcal{R}_\infty^{(i)}$ of $\tilde{\Lambda}^{(i)}$-mod, while the modules $M_C^{(i)}$ for $C \in k \setminus \{0,1\}$ lie in the mouths of the homogenous tubes $\mathcal{R}_C^{(i)}$.