ROOT CATEGORIES AND SIMPLE LIE ALGEBRAS

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1. Introduction

Let A be a finite dimensional associative hereditary algebra over some base field k which is representation-finite. In case k is algebraically closed, Gabriel [4] showed that the isomorphism classes of indecomposable A-modules correspond bijectively to the positive roots of the corresponding semisimple complex Lie algebra $\mathbf{g}$, and this result was extended to arbitrary base field by Dlab-Ringel [2]. It is natural to ask whether it is possible to recover a Lie algebra structure in terms of the representation theory of A. This was successfully attained by Ringel ([10],[11]). Let $K(\mathrm{mod}\, A)$ be the free abelian group with a basis indexed by the set of isomorphism classes of indecomposable A-modules. He showed that $K(\mathrm{mod}\, A)$ is the Chevalley $\mathbf{Z}$-form of $\mathbf{n}_+$ and hence $K(\mathrm{mod}\, A)\otimes_{\mathbf{Z}}\mathbf{C}$ and $\mathbf{n}_+$ are isomorphic Lie algebra, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of $!g$.

Ringel considered the case A over finite fields and filtrations of finite A-modules. He proved the existance of Hall polynomials for the filtrations and got an associative algebra which is called the Hall algebra of A, denoted by $\mathcal{H}_q(A)$, by using as structure constants the number of filtrations or evaluations of Hall polynomials at various values. $\mathcal{H}_1(A) \otimes \mathbf{C}$ is just the universal enveloping algebra of $K(\mathrm{mod}\, A) \otimes \mathbf{C}$ and $\mathcal{H}_1(A)$ is quite analogous to the Kostant $\mathbf{Z}$-form for a simple complex Lie algebra. Similar investigations were also given by Schofield [15] and Riedtmann [9] to consider the Euler characteristic of corresponding varieties instead of Hall polynomials.

Another advance in the representation theory of finite dimensional algebras is the theory of derived category due to Happel ([5],[6]). The derived category of a finite dimensional algebra B, denoted by $D^b(B)$, is a triangulated category with the translation functor T. In particular, the category $D^b(B)$ was well described when B is a tilted algebra.Happel pointed out that the isomorphism classes of indecomposable objects in $D^b(A)/T^2$ correspond bijectively to the root system of $\mathbf{g}$ (not only positive roots). For this reason, $\mathcal{R} = D^b(A)/T^2$ is called the root category. In [12], Ringel asked whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathbf{g}$ and not only $\mathbf{n}_+$. The present article gives an affirmative answer for this question by using the root category $\mathcal{R}$. Schofield has constructed the complete

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Lie algebra $\mathbf{g}$ too, in a different way [15]. But we deal with the simply-laced and non-simply-laced ! cases in a unified way and our met hod is computable.

Unfortunately we can not find an associative algebra related to $\mathcal{R}$ analoguous to the Hall algebra of A.We consider the triangles in $\mathcal{R}$ which are heritaged from $D^b(A)$, investigate the orbit space of all triangles of fixed type under the actions of certain automorphism groups, prove that the Hall polynomials exist for the cardinality of this orbit spaces, and use the evaluations of Hall polynomials as structrue constants to get a Lie product. Particular interesting to us is that the Cartan subalgebra $\mathbf{h}$ is realized by the weighted Grothendieck group of $\mathcal{R}$. We prove that $\mathbf{h}\oplus K(\mathcal{R})$ is a $\mathbf{Z}$-form of $\mathbf{g}$ and hence $(\mathbf{h}\oplus K(\mathcal{R}))\otimes_{\mathbf{Z}}\mathbb{C}\simeq\mathbf{g}$, where $K(\mathcal{R})$ is the free abelian group with a basis indexed by the set of isomorphism classes of indecomposable objects in $\mathcal{R}$. The isomorphism classes of indecomposable objects in $\mathcal{R}$ constitute a Chevalley basis of $\mathbf{g}$. In! some sense, our description for $\mathbf{g}$ may be considered as a global version of semisimple complex Lie algebras.

Although the definition of Lie product is given only in terms of the root category $\mathcal{R}$, if we compute $[u_M, u_N]$ for indecomposables M and N, except the case $N\simeq TM$ and then $[u_M,u_N]=H_M$, it coincides with that of M and N as modules of a hereditary algebra, which is defined by Ringel. In [10] Ringel gave a complete list of all nontrivial Hall polynomials which occur for a representation-finite hereditary algebra A. So the structure constants for the Lie product of $\mathbf{n}_+$ can be easily calculated. Our investigation implies that the calculation of Ringel actually provide all possibilities of Hall polynomials for $\mathcal{R}$ and of structure constants for the whole $\mathbf{g}$.

For unexplained notions concerning the representation theory of algebras we refer to [13], for those concerning the theory of Lie algebras to [7], and for the derived category to [6]

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2. Preliminaries

1. Let A be a finite dimensional hereditary k-algebra and let $D^b(A)$ be the derived category of mod A, which is a triangulated category with the translation functor T. Denote by $\mathcal{R}$ the orbirt category $D^b(A)/T^2$ and let $F: D^b(A)\longrightarrow\mathcal{R}$ be the canonical covering functor.

Lemma 2.1. For any indecomposible objects $M, N\in D^b(A)$, there exist $i,j\in\mathbf{Z}$ such that F induces the k-isomorphisms

$$\mathrm{Hom}_{D^b(A)}(T^iM,N)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{R}}(FM,FN)$$

and

$$\mathrm{Hom}_{D^b(A)}(M,T^jN)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{R}}(FM,FN).$$