Functors on locally finitely presented additive categories

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An additive category with direct limits is said to be locally finitely presented provided that the full subcategory of finitely presented objects is skeletally small and every object is a direct limit of finitely presented objects. Our aim in this paper is to study functors which are defined on locally finitely presented categories. Roughly, most of our results are of the following form. Given a category $\mathcal{C}$ and some property $(*)$, there exists a category $\mathcal{D}$ satisfying $(*)$ and a fully faithful functor $d\colon \mathcal{C} \to \mathcal{D}$ such that any functor $f\colon \mathcal{C} \to \mathcal{D}'$ into a category satisfying $(*)$ can be extended uniquely to a functor $f^*\colon \mathcal{D} \to \mathcal{D}'$ which respects $(*)$ and satisfies $f = f^* \circ d$. The following properties $(*)$ for a category $\mathcal{D}$ and combinations thereof are relevant:

(1) $\mathcal{D}$ has cokernels. (2) $\mathcal{D}$ has kernels. (3) $\mathcal{D}$ is abelian. (4) $\mathcal{D}$ has direct limits.

These "universal property" type results are used to study functors between locally finitely presented categories which have products or which are locally coherent, i.e. the full subcategory of finitely presented objects is abelian. For instance, we show that a functor $f\colon \mathcal{A} \to \mathrm{Ab}$ from a locally finitely presented category with products into the category of abelian groups commutes with direct limits and products if and only if there exists a presentation $\mathrm{Hom}(Y, \ ) \to \mathrm{Hom}(X, \ ) \to f \to 0$ such that X and Y are finitely presented. Also, we develop various criteria for the existence of left and right adjoints. Of particular interest is the following application. A locally finitely presented category is complete if and only if it is cocomplete. Much of this material can be found in the literature. However, the approach presented here appears to be new and will hopefully serve as a foundation for future work.

1. Preliminaries

We introduce some terminology. Throughout we are working in a fixed *universe* $\mathfrak{U}$ containing an infinite set (e.g. see [6, Numéro 0]). All categories $\mathcal{C}$ are assumed to be $\mathfrak{U}$-*categories* in the sense that for each pair of objects $X, Y \in \mathcal{C}$ the set $\mathrm{Hom}(X, Y)$ is *small*, i.e. bijective with a set in $\mathfrak{U}$. A category $\mathcal{C}$ is *skeletally small* provided that the isomorphism classes of objects in $\mathcal{C}$ form a small set. A non-empty category $\mathcal{I}$ is said to be *filtered* provided that for each pair of objects $\lambda_1, \lambda_2 \in \mathcal{I}$ there are morphisms $\varphi_i: \lambda_i \to \mu$ for some $\mu \in \mathcal{I}$, and for each pair of morphisms $\varphi_1, \varphi_2: \lambda \to \mu$ there is a morphism $\psi: \mu \to \nu$ with $\psi \circ \varphi_1 = \psi \circ \varphi_2$. We call the colimit $\varinjlim_{\lambda \in \mathcal{I}} X_\lambda$ of a functor $X: \mathcal{I} \to \mathcal{A}$, $\lambda \mapsto X_\lambda$ a *direct limit* if $\mathcal{I}$ is a skeletally small filtered category.

Given a category $\mathcal{A}$ with direct limits the finitely presented and finitely generated objects of $\mathcal{A}$ play an important role. Recall that an object $X \in \mathcal{A}$ is *finitely presented* (*finitely generated*) provided that for every direct limit $\varinjlim Y_\lambda$ in $\mathcal{A}$ the natural morphism $\varinjlim \operatorname{Hom}(X, Y_\lambda) \to \operatorname{Hom}(X, \varinjlim Y_\lambda)$ is an isomorphism (a monomorphism). The full subcategory of finitely presented objects of $\mathcal{A}$ is denoted by $\operatorname{fp}(\mathcal{A})$. Following [2], an additive category $\mathcal{A}$ is called *locally finitely presented* if $\operatorname{fp}(\mathcal{A})$ is skeletally small and every object in $\mathcal{A}$ is a direct limit of objects in $\operatorname{fp}(\mathcal{A})$.

Recall that an abelian category $\mathcal{A}$ is a *Grothendieck category* provided that $\mathcal{A}$ has coproducts, exact direct limits and a small generating set of objects. An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [1, Satz 1.5], [2, 2.4]. Now suppose that $\mathcal{A}$ is locally finitely presented and abelian. An object in $\mathcal{A}$ is finitely generated if and only if it is a quotient of some finitely presented object. The category $\mathcal{A}$ is said to be *locally coherent* provided that finitely generated subobjects of finitely presented objects are finitely presented, equivalently if $\operatorname{fp}(\mathcal{A})$ is abelian [8, Proposition 2.2].

Throughout the paper all functors between pre-additive categories are assumed to be additive. Given two pre-additive categories $\mathcal{C}$ and $\mathcal{D}$, the class of functors $F: \mathcal{C} \to \mathcal{D}$ is denoted by $(\mathcal{C}, \mathcal{D})$ and $\operatorname{Hom}(F, G)$ denotes the class of natural transformations between two functors F and G in $(\mathcal{C}, \mathcal{D})$. If $\mathcal{C}$ is skeletally small, then $(\mathcal{C}, \mathcal{D})$ forms actually a category since the Hom sets are small. The category of abelian groups is denoted by Ab. Given any category $\mathcal{C}$, one defines limits, colimits etc. in $(\mathcal{C}, \mathrm{Ab})$ pointwise and they coincide with the categorical notions if $(\mathcal{C}, \mathrm{Ab})$ is a category.

2. Finitely presented functors and abelian categories

In this section we collect some elementary facts about categories which have cokernels, kernels or which are abelian. Let $\mathcal{C}$ be a pre-additive category. A functor $F: \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is