QUASI-HEREDITARY ALGEBRAS AND TILTING MODULES

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Quasi-hereditary algebras, which were introduced by L.L.Scott [S], provide a connection between the representations in the sense of Lie theory and that in the sense of finite dimensional algebras, see [CPS]. V. Dlab and C. M. Ringel have made extensive studies to the ring and module theoretical approachs to this class of algebras. One of the most important results is the existence of the characteristic module, which was given by Ringel [R1] via Auslander-Reiten [AR]. This is both a tilting and cotilting module, and hence it gives a connection with tilting theory which has enjoyed a vigorous progress in the last decade.

This paper is organised as follows : we first fix some conventions and recall some preliminaries. By a result of Ringel, an algebra is quasi-hererditary if and only if there exists a tilting (or, cotilting) module having certain special properties, see [R1]; in the second section, we prove that a tilting module satisfying Ringel's original conditions must be the characteristic module, and hence it is unique; here we have used results of Coelho-Happel-Unger [CHU] on finding complements for partial tilting modules. Given a classical tilting triple (A, T, B), then B need not be quasi-hereditary, even if A is so (see [DR1]); in the third section we give a sufficient condition for B being quasi-hereditary. Finally, the so-called qht-algebras (one class of algebras which can be obtained from quasi-hereditary algebras after classical tilting functors, see section 4 below) are studied, we will see that this class of algebras has some similar properties to the class of quasi-hereditary algebras.

1. Preliminaries

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We assume that A is an artin algebra, $A-mod$ is the category of finitely generated left $A-$modules; $E(1), \cdots, E(n)$ is an ordering of simple A-modules, this amounts to an ordering of a complete set of orthogonal primitive idempotents $e_1, \cdots, e_n$ of A, which we simply denote by $\underline{\mathbf{e}} = (e_1, \cdots, e_n)$. Let $P(i)$ and $I(i)$ be the indecomposable projective and injective modules respectively with $topP(i) \cong E(i) \cong socI(i), 1 \leq i \leq n; \triangle = \{\triangle(1), \cdots \triangle(n)\}(resp. \bigtriangledown = \{\bigtriangledown(1), \cdots, \bigtriangledown(n)\})$ be the set of the standard (resp. costandard) modules, where $\triangle(i)$ is the maximal factor module of $P(i)$ with composition factors of the form $E(j)$ for $j \leq i$; and $\bigtriangledown(i)$ is the maximal submodule of $I(i)$ with composition factors of the form $E(j)$ for $j \leq i$. Denote by J_i the ideal $A(e_i + \cdots + e_n)A$ for $1 \leq i \leq n$. A is said to be quasi-hereditary with respect to the ordering $\underline{\mathbf{e}}$ provided first, ${}_AA \in \mathcal{F}(\triangle)$, and second, $E(i)$ occurs with multiplicity one in $\triangle(i)$ (or, equivalently, $End_A(\triangle(i))$ is a division ring) for $1 \leq i \leq n$. In the following we denote a quasi-hereditary algebra by a pair $(A, \underline{\mathbf{e}})$. Assume that $(A, \underline{\mathbf{e}})$ is quasi-hereditary, then there exists a uniquely defined basic tilting and cotilting module T such that $addT = \mathcal{F}(\triangle) \cap \mathcal{F}(\bigtriangledown)$, where $\mathcal{F}(\triangle)$ (resp. $\mathcal{F}(\bigtriangledown)$) is the full subcategory of $A-mod$ consisting of the modules having a filtration with the factors in $\triangle(resp. \bigtriangledown)$; and $addT$ is the additive category of direct sums of direct summands of T. Following [R1], T is then called the characteristic module of the quasi-hereditary algebra $(A, \underline{\mathbf{e}})$.

A module is said to be basic provided it has no direct summand of the form $N \oplus N$, with N non-zero. By definition an $A-$module T is a partial tilting module if T is of finite projective dimension and $Ext^i_A(T, T) = 0$ for $i \geq 1$; a partial tilting module T is said to be an almost complete tilting module provided T has $n-1$ indecomposable direct summands up to isomorphism, where n is the number of isomorphism classes of simple A-modules. A partial tilting module T is called a tilting module provided there exists an exact sequence $0 \longrightarrow_A A \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$ with all $T_i \in addT$. Dually, we have the notation of cotilting module. Let M be a partial tilting module, if N is an $A-$module such that $M \oplus N$ is a tilting module and $addM \cap addN = 0$, then N will be called a complement to M. A tilting (or, partial tilting) module with the projective dimension less or equal to 1 is called a classical tilting (resp. partial tilting) module. It is well known that a classical partial tilting module always admits a complement [B]. However, this is not true for partial tilting module in general, as pointed out in [RS]. Further, conditions for the existence and uniqueness of a complement to a partial tilting module were studied in [RS], [HU] and [CHU]. For a classical tilting triple (A,T,B)(i.e. T is a classical tilting A-module, $B = End_AT$), we denote by $(\mathcal{G}({}_AT), \mathcal{F}({}_AT))$ the torsion pair in $A - mod$ and by $(\mathcal{X}({}_AT), \mathcal{Y}({}_AT))$ the torsion pair in $B - mod$, both induced by T, see [HR].