$\mathcal{O}$ ALGEBRAS WITH THREE SIMPLE MODULES

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0. Introduction.

In [D], Dyer has introduced the concept of $\mathcal{O}$ algebra, this is a basic quasi-hereditary Koszul algebra with an anti-involution, such that its quadratic dual is also quasi-hereditary. Typical examples of such algebras are ones associated with blocks of category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand [BGG], see also Beilinson-Ginsburg-Soergel [BGS].

The present paper grew out of a desire to understand $\mathcal{O}$ algebras in the framework of the representation theory of finite-dimensional algebras, in particular, inside the one of quasi-hereditary algebras in the sense of Cline-Parshall-Scott [CPS]; see also Dlab-Ringel [DR] and Ringel [R2]. We consider such algebras with three (two) simple modules, this is the simplest and basic case: any $\mathcal{O}$ algebra has a subalgebra of this form.

We recall some definitions and facts in section 1. In section 2 we classify Dyer algebras with three simple modules in terms of quivers and relations (denoted by $\mathcal{O}(l,m,n)$), they are all quadratic. Representation types of $\mathcal{O}(l,m,n)$ are determined in section 3. It is proved in section 4 that algebras $\mathcal{O}(l,m,n)$ are closed under quadratic duals, and that they are all $\mathcal{O}$ algebras; as a corollary the global dimensions are given. In section 5 we give a sufficient condition for the existence of an exact Borel subalgebra (in the sense of König [K]) of a Dyer algebra, and then use this to compute Kazhdan-Lusztig-Vogan polynomials of $\mathcal{O}(l,m,n)$. One of the most important things in representations of a quasi-hereditary algebra is the existence of the characteristic module and the Ringel dual, see [R2]. In section 6 we discuss Ringel duals of $\mathcal{O}(l,m,n,M,B_v)$.

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1. Preliminaries.

1.1. Throughout the paper we consider basic finite-dimensional associative algebras over an algebraically closed field k. By a theorem of Gabriel such an algebra is given by quiver and relations.

Let A-mod denote the category of finite-dimensional left modules over A, and Λ the complete set of orthogonal primitive idempotents of A, which is assumed to be a poset and is called weight poset of A. If $i \in \Lambda$, denote by $L(i)$ the corresponding simple module; by $P(i)$ and $I(i)$ its projective cover and injective hull respectively, then the standard A-module $\Delta(i)$ (the costandard A-modules $\nabla(i)$, resp.) is defined to be the marximal factor module of $P(i)$ with composition factors $L(j)$ for $j \leq i$ (the marximal submodule of $I(i)$ with composition factors $L(j)$ for $j \leq i$, resp.); and A is said to be quasi-hereditary with weight poset Λ provided that first, $L(i)$ occurs with multiplicity one in $\Delta(i)$, and second, all projective A-modules are in $\mathcal{F}(\{\Delta(i), \forall i \in \Lambda\})$, where $\mathcal{F}(\{\Delta(i), \forall i \in \Lambda\})$ denotes the full subcategory of $A-$mod of modules having a filtration with all factors in $\{\Delta(i), \forall i \in \Lambda\}$, see [DR].

Assume that A is a quasi-hereditary algebra with weight poset Λ, then there exists a unique basic (generalized) tilting module T such that $addT = \mathcal{F}(\{\Delta(i), \forall i \in \Lambda\}) \cap \mathcal{F}(\{\nabla(i), \forall i \in \Lambda\})$, where $addT$ is the full subcategory of $A-$mod of direct sums of direct summands of T, and T is then called the characteristic module of (A, Λ); form the endomorphism algebra $B = End_A T$, then B is quasi-hereditary with weight poset Λ^{op}; and B is called the Ringel dual of A; note that the Ringel dual of B is exactly A; see [R2].

1.2. Let $A = k\vec{\triangle}/I$ with a finite quiver $\vec{\triangle}$ and an admissible ideal I, we identify the set of vertices of $\vec{\triangle}$ with the weight poset Λ. Let A_1 be the k-space spanned by arrows of $\vec{\triangle}$, denote by $A^{\pm}$ the subalgebra of A generated by the union $\Lambda \cup A_1^{\pm}$, where $A_1^+ = \sum\limits_{y>x} yA_1x$, and $A_1^- = \sum\limits_{y<x} yA_1x$. The following is called qh algebra in [D], here we use the name suggested by C.M.Ringel. Note that we don't require that A is a graded algebra.

Definition. Let A be an k-algebra with weight poset Λ, which satisfies the following conditions, then A is called a Dyer algebra with weight poset Λ:

(1) One has $A = A^- A^+$.

(2) One has $dim_k A \geq \sum\limits_{i \in \Lambda} (dim_k \Delta(i))^2$.

(3) There is an involutory algebra anti-automorphism ω of A such that $\omega(i) = i$ for $i \in \Lambda$ and ω sends arrows in $\vec{\triangle}$ to arrows.

Observe that if A is a Dyer algebra with weight poset Λ, then A is quasi-hereditary with Λ; in this case, the standard module $\Delta(i) = Ai/AA_1^+ i$ for $i \in \Lambda$;