A REALIZATION OF AFFINE LIE ALGEBRAS OF TYPE $\tilde{A}_{n-1}$ VIA THE DERIVED CATEGORIES OF CYCLIC QUIVERS

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Abstract. By using the locally finite-dimensional representations and the derived categories of a linear quiver $\Delta(\infty)$ and a cyclic quiver $\Delta(n)$, we give a realization of infinite rank affine Lie algebra A_∞ and affine Kac-Moody algebras of type $\tilde{A}_{n-1}$. Also their Chevalley bases are well displayed.

1. INTRODUCTION

According to the investigations of Ringel ([9],[10] etc.), there are Lie algebra structure arising from the module categories of a quite large class of representation-finite algebras. The most important case is that for a representation-finite hereditary algebra A, he showed that the Lie algebra $K(\text{mod } A)$ generated by indecomposable modules is isomorphic to the Chevalley $\mathbf{Z}$-form of $\mathbf{n}_+$, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of the corresponding complex semisimple Lie algebra $\mathbf{g}$. Now it is sure that Ringel's method is effective not only for representation-finite cases (see [1],[11]). However one should use the composition algebra to replace the Hall algebra in the representation-infinite cases.

Naturally the next question is that whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathbf{g}$ and not only $\mathbf{n}^+$. In [6] we gave a complete realization of complex semisimple Lie algebras in terms of the root categories. Denote by $D^b(B)$ the derived category of a finite dimensional algebra B, this is a triangulated category with the translation functor T. In particular, the category $D^b(B)$ was well described when B is a tilted algebra. Let A be a representation-finite hereditary algebra, Happel[2] pointed out that the isomorphism classes of indecomposable objects in $D^b(A)/T^2$ correspond bijectively to the root system of $\mathbf{g}$ (not only positive roots). For this reason, $\mathcal{R}(A) = D^b(A)/T^2$ is called the root category. Even when A is a representation-infinite hereditary algebra, the root category $\mathcal{R}(A)$ still provide a good framework which is comparable with the root system of the corresponding Kac-Moody algebras. We try to give the realization of affine Kac-Moody algebras by using the root category $\mathcal{R}(k\Delta)$, where Δ is the corresponding Euclidean

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quiver. In this paper, as the first step, we obtain the infinite rank affine Lie algebra A_∞ and affine Kac-Moody algebras of type $\tilde{A}_{n-1}$ from the root categories of locally finite dimensional representations of the linear quiver $\Delta(\infty)$ and the cyclic quiver $\Delta(n)$ respectively. At the same time we set up their Chevalley bases corresponding to the indecomposable objects in the root categories.

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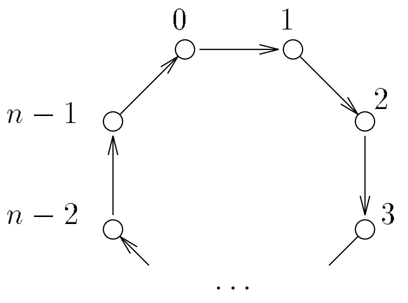
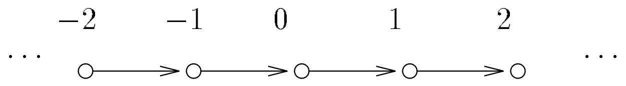
2. REPRESENTATIONS AND DERIVED CATEGORIES OF CYCLIC QUIVERS

A quiver $\Delta = (\Delta_0, \Delta_1) : \Delta_0$ is the set of vertices, Δ_1 the set of arrows. Let k be a field, a k-representation $V = (V_x, V_\alpha)$ of Δ is given by vector spaces V_x for all $x \in \Delta_0$ and linear maps $V_\alpha : V_x \to V_y$ for any arrow $\alpha : x \to y$. If V and V' are two representations of Δ, a map $f = (f_x) : V \to V'$ is given by maps $f_x : V_x \to V'_x$ for $x \in \Delta_0$ such that $f_x V'_\alpha = V_\alpha f_y$ for any arrow $\alpha : x \to y$. In this way we obtain the category of representations of Δ. A representation $V = (V_x, V_\alpha)$ of Δ is finite dimensional if $\sum_{x\in\Delta_0} dim_k V_x < \infty$. Denote by $rep(\Delta, k)$ the category of finite dimensional k-representations of Δ. In the present work we consider the finite dimensional representations of the cyclic quivers and the induced categories. A cyclic quiver $\Delta = \Delta(n)$ is the quiver



with $n \geq 1$; however maybe $n = \infty$, $\Delta(\infty)$ is the quiver



A stable translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ is given by a locally finite quiver (Γ_0, Γ_1), where Γ_0 denotes the vertex set and Γ_1 the set of arrows, together with a bijective map $\tau : \Gamma_0 \to \Gamma_0$ such that for any $y, z \in \Gamma_0$ the number of arrows from y to z is equal to the number of arrows from τz to y. In this article the arrows in Γ are