Elementary duality and local duality for modules

Henning Krause

Given a ring Λ (associative with identity) we denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules and by $\mathrm{mod}(\Lambda)$ the full subcategory of all finitely presented Λ-modules. Our aim in this note is to point out a relation between elementary duality for modules as introduced by Herzog [6] and a classical construction in module theory. We call an indecomposable pure-injective Λ-module *simply reflexive* provided that there exists a map $X \to Y$ in $\mathrm{mod}(\Lambda)$ such that the cokernel of the induced map $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M)$ is simple when it is regarded in the natural way as an $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module. Given such a module M it has been shown in [8] that there exists, up to isomorphism, a unique indecomposable pure-injective Λ^{op}-module DM such that any map φ in $\mathrm{mod}(\Lambda)$ induces an epimorphism $\mathrm{Hom}_\Lambda(\varphi, M)$ if and only if it induces a monomorphism $\varphi \otimes_\Lambda DM$. Moreover, DM is again simply reflexive and satisfies $DDM = M$. Note that M is simply reflexive if and only if it is reflexive in the sense of Herzog [6] and that DM coincides with the dual of M in the sense of Herzog [6]. In this note we shall prove the following result.

Theorem *Let M be an indecomposable pure injective Λ-module and suppose that M is simply reflexive. Suppose also that M is a Λ-Γ-bimodule and that I is an injective cogenerator for* $\mathrm{Mod}(\Gamma)$. *Then DM is isomorphic to a direct summand of the Λ^{op}-module* $\mathrm{Hom}_\Gamma(M, I)$.

Remark (1) The proof of the theorem shows how to construct DM.

(2) For examples of simply reflexive modules we refer to [6]. For instance, any Σ-pure-injective module is simply reflexive.

(3) Let $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ and regard M in the natural way as a Γ-module. Taking a minimal injective cogenerator I the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ might be called the *local dual* of M. It is well-known that $\mathrm{Hom}_\Gamma(M, I)$ is indecomposable if M is finitely presented. If M is of finite length as a Γ-module, then it has been shown by Crawley-Boevey [3] that $\mathrm{Hom}_\Gamma(M, I)$ is a coproduct of copies of DM.

(4) Recall from [8] that the *endocategory* $\mathcal{E}_M$ of M is the smallest abelian subcategory of $\mathrm{Mod}(\mathrm{End}_\Lambda(M)^{\mathrm{op}})$ containing M and all the endomorphisms of M induced by multiplication with an element from Λ. The modules M, DM and $\mathrm{Hom}_\Gamma(M, I)$ are related by a duality between $\mathcal{E}_M$ and $\mathcal{E}_{DM}$ and an equivalence between $\mathcal{E}_{DM}$ and $\mathcal{E}_{\mathrm{Hom}_\Gamma(M,I)}$.

To give a proof of the theorem we need to recall some background material. Let $\mathcal{C}_\Lambda = \operatorname{mod}(\Lambda^{\mathrm{op}})^{\mathrm{op}}$. We denote by $\operatorname{Mod}(\mathcal{C}_\Lambda)$ the category of all additive functors $(\mathcal{C}_\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups and by $\operatorname{mod}(\mathcal{C}_\Lambda)$ the full subcategory of all finitely presented functors which is abelian. The fully faithful functor

$$\operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. An exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{Mod}(\Lambda)$ is said to be *pure-exact* if its image

$$0 \longrightarrow L \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow 0$$

under this functor is exact and $M \in \operatorname{Mod}(\Lambda)$ is *pure-injective* if $M \otimes_\Lambda -$ is injective. Given a module M we denote by $\mathcal{S} = \mathcal{S}_M$ the kernel of the functor

$$\operatorname{mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\Gamma), \quad X \mapsto \operatorname{Hom}(X, M \otimes_\Lambda -)$$

where $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ is identified with $\operatorname{End}(M \otimes_\Lambda -)^{\mathrm{op}}$. Furthermore, denote by $\vec{\mathcal{S}}$ the full subcategory of $\operatorname{Mod}(\mathcal{C}_\Lambda)$ which consists of all direct limits $\varinjlim X_i$ with $X_i \in \mathcal{S}$ for all i.

Recall that a full subcategory $\mathcal{T}$ of any module category $\operatorname{Mod}(\mathcal{C})$ is *localizing* if it is closed under subobjects, quotients, extensions and coproducts. For any localizing subcategory $\mathcal{T}$ one can form the *quotient category* $\operatorname{Mod}(\mathcal{C})/\mathcal{T}$ which is abelian, has injective envelopes and admits an exact *quotient functor* $q\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Mod}(\mathcal{C})/\mathcal{T}$ with $\operatorname{Ker}(q) = \mathcal{T}$ [4].

Lemma 1 *The subcategory $\vec{\mathcal{S}}$ is localizing. The quotient functor $q\colon \operatorname{Mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}$ sends any injective object N satisfying $\operatorname{Hom}(\mathcal{S}, N) = 0$ to an injective object and q induces an isomorphism $\operatorname{Hom}(X, N) \to \operatorname{Hom}(q(X), q(N))$ for every $X \in \operatorname{Mod}(\mathcal{C}_\Lambda)$.*

Proof: The first statement is proved in [7]. The properties of q are well-known facts which may be found in [4].

Lemma 2 *Let $\varphi\colon X \to Y$ be a map in $\operatorname{mod}(\Lambda)$ and let $U = \operatorname{Ker}(\varphi \otimes_\Lambda -)$ in $\operatorname{Mod}(\mathcal{C}_\Lambda)$. Then $\operatorname{Hom}(U, M \otimes_\Lambda -) \cong \operatorname{Coker}(\operatorname{Hom}_\Lambda(\varphi, M))$ for any $M \in \operatorname{Mod}(\Lambda)$.*

Proof: The exact sequence $0 \to U \to X \otimes_\Lambda - \to Y \otimes_\Lambda -$ induces an exact sequence

$$\operatorname{Hom}(Y \otimes_\Lambda -, M \otimes_\Lambda -) \to \operatorname{Hom}(X \otimes_\Lambda -, M \otimes_\Lambda -) \to \operatorname{Hom}(U, M \otimes_\Lambda -) \to 0.$$