The endocategory of a module

Henning Krause

Dedicated to the memory of Maurice Auslander

Given a ring Λ we introduce the endocategory $\mathcal{E}_M$ of a Λ-module M. It is an abelian subcategory of $\mathrm{Mod}(\Gamma)$ where $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$, and $\mathcal{E}_M$ is constructed in such a way that it is the smallest abelian subcategory of $\mathrm{Mod}(\Gamma)$ containing M regarded in the natural way as a Γ-module and all the endomorphisms of M induced by multiplication with an element from Λ. The first aim of this paper is to discuss some basic properties of this category. For instance, we show that $\mathcal{E}_M$ reflects various properties of the module M which are related to purity. Another aim is to give a functorial description of the Ziegler spectrum of Λ which is, by definition, a representative set of indecomposable pure-injective Λ-modules, together with a topology introduced by Ziegler [14]. We shall also discuss the relation between certain right and left Λ-modules which arises from the well-known duality between the categories of finitely presented functors from $\mathrm{mod}(\Lambda^{\mathrm{op}})$ and $\mathrm{mod}(\Lambda)$, respectively, into the category of abelian groups. In fact, this duality induces a bijection $M \mapsto DM$ between certain subsets of the Ziegler spectra of Λ and Λ^{op}, respectively, which has been studied by Herzog [6] using positive primitive formulas, and by Crawley-Boevey [1] using characters. The functoriality of this bijection is expressed by dualities between the endocategories $\mathcal{E}_M$ and $\mathcal{E}_{DM}$. A final example illustrates these dualities as well as their limitations.

This paper is dedicated to the memory of Maurice Auslander. In fact, the material presented here depends in an essential way on homological and categorical techniques and ideas that indelibly bear his mark.

1. Preliminaries

Let $\mathcal{C}$ be a skeletally small pre-additive category. A *(right) $\mathcal{C}$-module* is an additive functor from $\mathcal{C}^{\mathrm{op}}$ into the category Ab of abelian groups and we denote by $\mathrm{Mod}(\mathcal{C})$ the category of all $\mathcal{C}$-modules. Recall that $M \in \mathrm{Mod}(\mathcal{C})$ is said to be *finitely presented* provided that there exists an exact sequence $\coprod_{i=1}^n \mathrm{Hom}(\ ,X_i) \to \coprod_{j=1}^m \mathrm{Hom}(\ ,Y_j) \to M \to 0$ in $\mathrm{Mod}(\mathcal{C})$, and M is *finitely generated* if it is a quotient of a finitely presented module. The full subcategory of all finitely presented $\mathcal{C}$-modules is denoted by $\operatorname{mod}(\mathcal{C})$. It is an additive catgory with cokernels, and $\mathcal{C}$ is called *coherent* if $\operatorname{mod}(\mathcal{C})$ is abelian. Given a ring Λ we shall view Λ as a category with one object and we define $\mathcal{C}_\Lambda = \operatorname{mod}(\Lambda^{\mathrm{op}})^{\mathrm{op}}$. Note that this category is coherent. The fully faithful functor

$$\operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. An exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{Mod}(\Lambda)$ is said to be *pure-exact* if its image

$$0 \longrightarrow L \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow 0$$

under this functor is exact and $M \in \operatorname{Mod}(\Lambda)$ is *pure-injective* if $M \otimes_\Lambda -$ is injective. Finally, recall that there is a well-known duality $d\colon \operatorname{mod}(\mathcal{C}_\Lambda) \to \operatorname{mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})$ given by

$$d(M)(X) = \operatorname{Hom}(M, X \otimes_\Lambda -).$$

Thus $\operatorname{mod}(\mathcal{C}_\Lambda)$ has sufficently many projectives which are the functors $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(Y, \ )$, $Y \in \operatorname{mod}(\Lambda^{\mathrm{op}})$, and sufficently many injectives which are the functors $X \otimes_\Lambda -$, $X \in \operatorname{mod}(\Lambda)$ [5].

We continue with some general facts. Recall that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* provided that for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ the object Y belongs to $\mathcal{S}$ iff X and Z belong to $\mathcal{S}$. For any Serre subcategory $\mathcal{S}$ of $\mathcal{A}$ one can form the *quotient category* $\mathcal{A}/\mathcal{S}$ which is abelian and admits an exact *quotient functor* $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ with $\operatorname{Ker}(q) = \mathcal{S}$ [4]. The functor q is characterized by the property that any exact functor $f\colon \mathcal{A} \to \mathcal{B}$ between abelian categories with $\operatorname{Ker}(f) = \mathcal{S}$ induces, up to isomorphism, a unique faithful and exact functor $\bar{f}\colon \mathcal{A}/\mathcal{S} \to \mathcal{B}$ such that $f = \bar{f} \circ q$. The subcategory $\mathcal{S}$ is called *localizing* if q has a right adjoint.

Suppose now that $\mathcal{C}$ is a skeletally small pre-additive category which is coherent, e.g. $\mathcal{C} = \mathcal{C}_\Lambda$. Let $\mathcal{S}$ be a Serre subcategory of $\operatorname{mod}(\mathcal{C})$ and denote by $\vec{\mathcal{S}}$ the full subcategory of $\operatorname{Mod}(\mathcal{C})$ which consists of all direct limits $\varinjlim X_i$ with $X_i \in \mathcal{S}$ for all i.

Lemma 1.1 *The subcategory $\vec{\mathcal{S}}$ is localizing and the quotient functor $q\colon \operatorname{Mod}(\mathcal{C}) \to \operatorname{Mod}(\mathcal{C})/\vec{\mathcal{S}}$ has the following properties.*

(1) *Let $M \in \operatorname{Mod}(\mathcal{C})$ with $\operatorname{Hom}(\mathcal{S}, M) = 0$. Then M is injective iff $q(M)$ is injective. In that case q induces an isomorphism $\operatorname{Hom}(X, M) \to \operatorname{Hom}(q(X), q(M))$ for every $X \in \operatorname{Mod}(\mathcal{C})$.*