ON THE EXISTENCE OF POSTPROJECTIVE COMPONENTS IN THE AUSLANDER-REITEN QUIVER OF AN ALGEBRA

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Let k be an algebraically closed field and A be a basic finite-dimensional k-algebra of the form $A = kQ/I$, where Q is a quiver (= finite oriented graph) and I is an admissible ideal of the path algebra kQ, see [3]. In this work we assume that Q has no oriented cycles.

Let mod_A denote the category of finite-dimensional left A-modules. For each indecomposable non-projective A-module X, the Auslander-Reiten translate $\tau_A X$ is an indecomposable non-injective module. The Auslander-Reiten quiver Γ_A has as vertices representatives of the isoclasses of the finite-dimensional indecomposable A-modules, there are as many arrows from X to Y as $\dim_k \mathrm{rad}_A(X,Y)/\mathrm{rad}^2_A(X,Y)$. In this paper we do not distinguish between a module and its corresponding isoclass. A connected component $\mathcal{P}$ of Γ_A is *postprojective* if $\mathcal{P}$ has no oriented cycles and each module X in $\mathcal{P}$ has only finitely many predecessors in the path order of $\mathcal{P}$. Several important classes of algebras have postprojective components: hereditary algebras [3, 6], algebras satisfying the separation condition [1, 2], tilted algebras [8].

The aim of this work is to find necessary and sufficient conditions for the existence of postprojective components in Γ_A. In section 1 we give an algorithmic procedure to decide the existence of postprojective components. In section 2 we consider a one-point extension algebra $A = B[M]$ such that all indecomposable direct summands of M belong to postprojective components of Γ_B, then we give conditions that assure that the projective A-module P with $\mathrm{rad}\, P = M$ lies in a postprojective component of Γ_A. In section 3 we consider some special cases. We recall that once identified a postprojective component $\mathcal{P}$ of Γ_A, the modules on $\mathcal{P}$ may be constructed using the *knitting procedure* [3]. In [5], an algorithmic procedure which makes essential use of the knitting procedure is given to construct all the postprojective components of Γ_A.

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1. Existence of postprojective components.

1.1. Let $A = kQ/I$ be a finite-dimensional k-algebra such that the quiver Q has no oriented cycles. We may consider A as a k-category with objects the set of vertices Q_0 of Q and morphisms from $x, y \in Q_0$

the space $A(x,y) = e_y A e_x$, where e_x denotes the trivial path at the vertex x. For two vertices $x, y \in Q_0$ we write $y \leq x$ if there is a path from y to x in Q.

Let $x \in Q_0$, we denote by A^x the full subcategory of A whose vertices are those $y \in Q_0$ with $y \not\leq x$. Observe that the quiver Q^x of A^x is a convex (= path closed) subquiver of Q. The indecomposable projective A-module $P_x = Ae_x$ has radical $\operatorname{rad} P_x$ which is an A^x-module. We denote by $\operatorname{rad} P_x = \bigoplus_{i=1}^{n_x} R_i^x$ the indecomposable decomposition of $\operatorname{rad} P_x$.

1.2. A *path* in mod_A is a sequence $(X_0, \ldots, X_s)$ of (isomorphisms classes of) indecomposable A-modules X_i, $0 \leq i \leq s$, such that there is a map $0 \neq f_i \in \operatorname{Hom}_A(X_i, X_{i+1})$ which is not an isomorphism, $0 \leq i \leq s-1$. In this case we write $X_0 \leq X_s$ and we say that X_0 is a *predecessor* of X_s. If $s = 1$ and $X_0 = X_s$ we say that the path $(X_0, \ldots, X_s)$ is a *cycle*.

Following [4] we say that a module M is *directing* in mod_A provided there do not exist indecomposable direct summands M_1 and M_2 of M and an indecomposable non-projective module X such that $M_1 \leq \tau X$ and $X \leq M_2$. It is shown in [4] that an indecomposable module X is directing if and only if there are no cycles $(X_0, \ldots, X_s)$ with $X_0 = X = X_s$. The following result will be important in our work.

Theorem [4, 7]. *Let $x \in Q_0$. Then P_x is directing in mod_A if and only if $\operatorname{rad} P_x$ is directing in mod_A. Moreover, if x is a source, then P_x is directing in mod_A if and only if $\operatorname{rad} P_x$ is directing in mod_{A^x}.*

1.3. We state our main result which provides an algorithmic criterion for the existence of postprojective components.

Theorem. *Let $A = kQ/I$ be a finite-dimensional k-algebra such that Q has no oriented cycles. Then Γ_A has a postprojective component if and only if for each vertex $x \in Q_0$ one of the following conditions is satisfied:*

(1x) *there is a postprojective component $\mathcal{P}$ of Γ_{A^x} such that $R_i^x \notin \mathcal{P}$ for every $1 \leq i \neq n_x$;*

(2x) *for each $1 \leq i \leq n_x$ the set of predecessors $\{Y \in \Gamma_{A^x} : Y \leq R_i^x\}$ of R_i^x in mod_{A^x} is finite and formed by directing modules. Moreover, if x is a source, then $\operatorname{rad} P_x$ is directing in mod_{A^x}.*

We prove the theorem in (1.5) after some preparation. In (1.8) we give some examples.

1.4. Lemma. *Assume that for all $x \in Q_0$ the condition (2x) is satisfied, then Γ_A has a postprojective component.*

Proof: We claim that for every $x \in Q_0$ the following condition is satisfied:

(3x): for each $1 \leq i \leq n_x$, the set of predecessors $\{X \in \Gamma_A : X \leq R_i^x\}$ of R_i^x in mod_A is finite and formed by directing modules.