Arnoldi Type Algorithms for Large Unsymmetric Multiple Eigenvalue Problems*

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Abstract

Two Arnoldi type algorithms are proposed for large unsymmetric *multiple* eigenvalue problems when the matrix A involved is *diagonalizable*. The theoretical background is established. The algorithms can adaptively determine the multiplicity of a eigenvalue and a basis of the associated *eigenspace*. Numerical experiments show reliability of the algorithms.

Keywords. multiple eigenvalue, diagonalizable, Arnoldi's algorithm, eigenprojector, convergence, error bounds

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1 Introduction

The Lanczos algorithm is a very powerful tool for extracting a few extreme eigenvalues and associated eigenvectors of large symmetric matrices [9, 11]. Since the 1980's, considerable attention has been paid to generalizing it to large unsymmetric problems. One of its generalizations is Arnoldi's method [1, 14]. It can be used to compute outer part of the spectrum and corresponding eigenvectors [6, 7, 8, 14, 15, 17]. In order to improve overall performance, Saad [16] suggests to use it in conjunction with the Chebyshev iteration. There are other variants available; see, e.g. [13, 17].

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To apply Arnoldi's algorithm and its variants to practical problems, one must account for some difficulties which can arise in applications [2, 3], e.g.

Difficulty* *Multiple eigenvalues are a common occurrence.*

In the symmetric case, Parlett *et al.* [10] use the Lanczos algorithm with selective orthogonalization to solve Difficulty*. It maintains the semi-orthogonality among the Lanczos vectors so as to avoid the occurrence of spurious eigenvalues and determines the multiplicities of the required eigenvalues and the associated eigenspaces by restarting. Its key idea is that, before restarting, a new initial vector is orthogonalized with respect to all the converged eigenvectors until the eigenspace associated with a multiple eigenvalue is found.

In the unsymmetric case, the situation becomes much more complicated. The strategy of restarting [10] cannot solve Difficulty* here, since the eigenvectors of unsymmetric matrices are, in general, not mutually orthogonal just as those of symmetric matrices are. The mutual orthogonality of eigenvectors forms the basis of the algorithm in [10]. Theoretically speaking, a simple simulation of the idea used in [10] suggests that before restarting we use Arnoldi's method with a new initial vector orthogonal to all the *left* eigenvectors of the matrix A associated with all the converged right eigenvectors. Proceeding in such a way, we can find the multiplicities of the required eigenvalues and determine the associated eigenspaces. However, an easy analysis [12] shows that Arnoldi's method is inefficient for computing the left eigenvectors of A. Of course, one can apply Arnoldi's method to A^H, the conjugate transpose of A, to get the left eigenvectors of A, while this doubles the amount of computation.

In order to deal with Difficulty*, generalized block Lanczos methods are studied in [7, 8]. They can be used to compute outer part of the spectrum and corresponding eigenvectors, up to a multiplicity equal to the block size when A is diagonalizable. However, if the multiplicities of the required eigenvalues are bigger than the block size, the block algorithms themselves are not able to determine the multiplicity of a multiple eigenvalue and the associated eigenspace.

In this paper, we attempt to design Arnoldi type algorithms for solving such a Difficulty* when A is *diagonalizable*. As is seen from [7, 8], although the idea proposed seems not complicated, it is important not only in its own right but also indispensable for the block Arnoldi methods when the block size is less than or equal to the multiplicities of the required eigenvalues.

In Section 2, we introduce the notation used and go through the underlying Arnoldi algorithm; in Section 3, assuming that A is diagonalizable, we present the theoretical background of the Arnoldi type algorithms proposed in Section 4. Some of the results, i.e. theoretical error bounds for eigenvectors, are also new for both Arnoldi's method and the perturbation analysis of a general eigenvalue problem; in Section 4 we present two Arnoldi type algorithms to solve Difficulty* for the diagonalizable A, the second of which is the first with the Chebyshev iteration [5, 16]; in Section 5, we discuss implementations of the algorithms; in Section 6, we report on several numerical examples to