Z–matrices and inverse Z–matrices

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Abstract

We consider Z–matrices and inverse Z–matrices, i.e. those nonsingular matrices, whose inverse is a Z–matrix. Recently Fiedler and Markham introduced a classification of Z–matrices. This classification directly leads to a classification of inverse Z–matrices. Among all classes of Z–matrices and inverse Z–matrices the classes of M–matrices, N_0–matrices, F_0–matrices and inverse M–matrices, inverse N_0–matrices and inverse F_0–matrices respectively, had been studied in detail. Here we discuss each single class of Z–matrices and inverse Z–matrices as well as we consider the whole classes of Z–matrices and inverse Z–matrices. We establish some common properties of the classes like eigenvalue bounds and determinant inequalities and we give a new characterization of all classes of Z–matrices and inverse Z–matrices. Some of the results generalize known results for M–matrices, N_0–matrices, and F_0–matrices, and inverse M–matrices, inverse N_0–matrices, and inverse F_0–matrices. However, we will also show that some properties of the specific classes mentioned above do not hold for all classes of Z–matrices and inverse Z–matrices.

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1 Introduction and Notation

Throughout we deal with $n \times n$ Z–matrices, i.e. real matrices whose off-diagonal entries are nonpositive. These matrices arise in many problems in the mathematical and physical sciences. Some of the best-known subclasses of Z–matrices are the class of M–matrices (introduced by Ostrowski), the class of N_0–matrices (introduced by Ky Fan and G.A. Johnson) and the class of F_0–matrices (introduced by G.A. Johnson). Especially for M–matrices a large amount of properties and characterizations exists. However, the other classes of Z–matrices are also of great interest. The first systematical effort in considering the whole class of Z–matrices was made by Fiedler and Markham in [FM]. They introduced a classification of Z–matrices by generalizing the ideas of the definitions of M–matrices, N_0–matrices and F_0–matrices and gave some common properties of all classes. Here we continue discussing the whole class of Z–matrices as well as considering all subclasses of Z–matrices. We summarize some known results and establish new results like a new characterization of all classes of Z–matrices, eigenvalue bounds and determinant inequalities.

On the other hand, there has been interest in inverse Z–matrices, i.e. any nonsingular matrices whose inverse is a Z–matrix. The classification of Z–matrices directly leads to a classification of inverse Z–matrices. All matrices, whose inverse belongs to the same class of Z–matrices, build a subclass of the class of all inverse Z–matrices. The problem to determine whether or whether not a matrix is an inverse Z–matrix or more preciously, is an inverse Z–matrix of a specific class, is called the inverse Z–matrix problem. This problem contains the inverse M–matrix problem, i.e. characterizing all (nonnegative) matrices whose inverse is an M–matrix. Similar to the M–matrix case, there exists an amount of papers considering inverse M–matrices. A survey of this topic is given by C.R. Johnson [J1]. However a complete solution of this problem is still missing. Also inverse N_0–matrices and inverse F_0–matrices has been considered, e.g. G.A. Johnson [J3], Smith [S2], and Ying Chen [C]. The first systematically approach in considering all classes of inverse Z–matrices was made by Nabben and Varga [NV3]. There the authors gave sufficient conditions for matrices to be an inverse Z–matrix for each specific class. Here we give new results for the whole class of inverse Z–matrices as well as for each single subclass. We establish a new characterization and give some new results of the structure of an inverse Z–matrix. Moreover, we give a new set of sufficient conditions of all classes of inverse Z–matrices.

The paper is organized as follows. The next section deals with Z–matrices. This section includes the classification of Z–matrices, which we also need in Section 3. In Section 3 we consider inverse Z–matrices. At the end of the section we give examples of inverse Z–matrices, the so-called type D matrices, which will be studied in detail.

We will use the following notation:
By I_r we denote the $r \times r$ identity matrix. The index r is omitted if $r = n$. If $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is nonnegative, i.e. $a_{ij} \geq 0$, we write $A \geq 0$. Similarly, we write $A \geq B$, for $A, B \in \mathbb{R}^{n,n}$, if and only if $A - B \geq 0$. The spectral radius of $A \in \mathbb{R}^{n,n}$ is denoted by $\rho(A)$. The vector