Constructing large modules over artin algebras

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Let Λ be an artin algebra and denote by $\operatorname{mod}(\Lambda)$ the category of finitely generated Λ-modules. Apart from those we are also interested in Λ-modules which are not finitely generated. These are called large. Recall that the algebra Λ is of infinite representation type provided that $\operatorname{mod}(\Lambda)$ has infinitely many isomorphism classes of indecomposable objects. A theorem of Auslander asserts that there exists a large indecomposable Λ-module whenever Λ is of infinite representation type [2]. In this note we use his method to construct specific large indecomposable modules which arise naturally from certain infinite families of morphisms in $\operatorname{mod}(\Lambda)$ having isomorphic kernels. We apply this result as follows. Suppose there is given a chain

$$X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} \ldots \xrightarrow{\psi_{s-1}} X_s \xrightarrow{\psi_s} \ldots$$

of monomorphisms between finitely generated indecomposable Λ-modules. Denote by $X_\infty = \varinjlim X_i$ the corresponding direct limit. It is natural to ask for conditions that ensure the indecomposability of X_∞. In fact, chains of monomorphisms of the above form occur for quasi-serial components of the Auslander-Reiten quiver of Λ. Ringel calls any module $X = X_s$ belonging to such a component quasi-serial of quasi-length s, since it determines, up to isomorphism, uniquely a chain of irreducible monomorphisms $\psi_i\colon X_i \to X_{i+1}$, $i \in \mathbb{N}$, such that there exists no irreducible monomorphism ending in X_1 [8]. We show that the corresponding direct limit $X_\infty = \varinjlim X_i$ is a large indecomposable module. Moreover, we obtain an exact sequence

$$0 \longrightarrow X \xrightarrow{\psi_X} X_\infty \longrightarrow (\operatorname{Tr} D)^s(X)_\infty \longrightarrow 0$$

where $(\operatorname{Tr} D)^s(X)$ denotes the s-th power of the transpose of the dual of X which is again a quasi-serial module. This exact sequence has the following property. Every morphism $\theta\colon X \to Y$ in $\operatorname{mod}(\Lambda)$ belongs to the infinite radical of $\operatorname{mod}(\Lambda)$ if and only if θ factors through ψ_X, and moreover, θ induces a commutative diagram of the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\psi_X} & X_\infty & \longrightarrow & (\operatorname{Tr} D)^s(X)_\infty & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\psi_Y} & Y_\infty & \longrightarrow & (\operatorname{Tr} D)^t(Y)_\infty & \longrightarrow & 0
\end{array}$$

provided that Y is quasi-serial of quasi-length t.

We would like to point out as an application that modules of the form X_∞ can be used to construct generic modules in the sense of Crawley-Boevey [5]. This will be discussed in a forthcoming publication. Note that the existence of a generic module is crucial for the validity of the second Brauer-Thrall conjecture for artin algebras [5].

The construction

Let Λ be an artin algebra and denote by $\mathrm{Mod}(\Lambda)$ the category of right Λ-modules. Given two morphisms $\varphi_i: X \to Y_i$, $i = 1, 2$, in $\mathrm{Mod}(\Lambda)$ we define $\varphi_2 \preceq \varphi_1$ if there exists a morphism $\psi: Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_1$. We write $\varphi_2 \prec \varphi_1$ if $\varphi_2 \preceq \varphi_1$ and $\varphi_1 \not\preceq \varphi_2$. Recall that a morphism $\varphi: X \to Y$ in $\mathrm{mod}(\Lambda)$ is *irreducible* provided that φ is neither a split monomorphism nor a split epimorphism and if for any factorization $\varphi = \varphi_2 \circ \varphi_1$ in $\mathrm{mod}(\Lambda)$ either φ_1 is a split monomorphism or φ_2 is a split epimorphism.

Theorem 1 *Let $0 \to X \xrightarrow{\varphi_i} Y_i \xrightarrow{\chi_i} Z_i \to 0$, $i \in \mathbb{N}$, be a family of exact sequences in* $\mathrm{mod}(\Lambda)$*. Suppose that χ_i is irreducible and that $\varphi_{i+1} \prec \varphi_i$ for all $i \in \mathbb{N}$. Then there exists a family of morphisms $\psi_i: Y_i \to Y_{i+1}$ satisfying $\varphi_{i+1} = \psi_i \circ \varphi_i$ for all $i \in \mathbb{N}$ such that $Y = \varinjlim Y_i$ is a large denumerably generated indecomposable module.*

We postpone a proof and discuss some preliminaries. We shall use the category $(\mathrm{mod}(\Lambda), \mathrm{Ab})$ of additive functors from $\mathrm{mod}(\Lambda)$ to the category Ab of abelian groups. Note that for a pair of morphisms $\varphi_i: X \to Y_i$, $i = 1, 2$, in $\mathrm{mod}(\Lambda)$ we have $\varphi_2 \preceq \varphi_1$ iff $\mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_2, \ )) \subseteq \mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_1, \ ))$ as subfunctors of $\mathrm{Hom}_\Lambda(X, \ )$. We shall also use the fully faithful functor

$$\mathrm{Mod}(\Lambda) \to (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad X \mapsto X \otimes_\Lambda -$$

which identifies the pure-injective Λ-modules with the injective objects in $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. Note that $\mathrm{Ext}^1(\ , X \otimes_\Lambda -)$ vanishes on finitely presented functors for every Λ-module X. Moreover, every finitely generated Λ-module is automatically pure-injective. The following characterization of an irreducible morphism in $\mathrm{mod}(\Lambda)$ is needed.

Lemma 2 *For a non-split exact sequence $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\chi} Z \to 0$ in* $\mathrm{mod}(\Lambda)$ *the following are equivalent.*