VISUAL BASIC REPRESENTATIONS: AN ATLAS

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Abstract. We depict the weight diagrams of basic and adjoint representations of complex simple Lie algebras/algebraic groups and describe some of their uses.

Introduction

In this paper we collect the weight diagrams of basic representations of complex simple Lie algebras as well as of those adjoint representations, which are not basic. These pictures arise in a number of contexts, but their main significance stems from the fact that they allow to visualize calculations with root systems, Weyl

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groups, Lie algebras, Chevalley groups and their representations, to a large extend replacing (or sometimes enhancing) such tools as calculations with matrices or Bruhat decompositions.

These pictures and related combinatorial objects appeared dozens (hundreds?) of times in various contexts, such as representation theory and structure theory of semisimple Lie algebras, algebraic groups and Lie groups, invariant theory, algebraic K-theory, combinatorial geometries, Schubert calculus, Jordan systems, Hermitian symmetric spaces, combinatorics, computer algebra, etc. Our primary emphasis in this paper are the pictures themselves, rather than their uses. In this sense it is a *pendant* to [V7], which contains a detailed description of significance of the pictures and various numerical invariants connected with them, as well as proofs of some results quoted here *en passant*.

The paper is organized as follows. In Section 1 we recall the notation. In Section 2 we define the weight diagrams, discuss various ways to draw them and explain how the diagrams in the atlas were constructed. Finally, Section 3 discusses some of the applications of the pictures referring to [BE], [BC1], [BC2], [CCu], [Cl], [CIK], [DES], [ES1], [ES2], [HS], [Hi1], [Hi2], [Ho], [Ir], [Mz2], [PR], [Pl3] – [Pl5], [Sch], [St2], [V2], [V4], [V6] – [V9], [VP1] – [VP3], [Z] for a detailed explanation of these and further examples and many additional references. In the bibliography we cite further papers containing these and similar pictures. Tables 1 and 2 reproduce the numbering of the fundamental roots and the list of basic and adjoint representations. The core of the paper, its *raison d'être*, are the pictures themselves, Figures 1 – 28.

1. Basic notions

In this section we briefly recall the notation used in the sequel and the notion of a basic representation. All background information on root systems, Lie algebras, algebraic groups, Chevalley groups and representations may be found in [Bo1], [Bo2], [Bu1], [Bu2], [Ca], [Hé], [H1] – [H3], [Ja], [Sp1], [Sp2], [Sb].

1°. Root systems and Weyl groups. Let Φ be a reduced irreducible root system of rank l, $Q(\Phi)$ be the root lattice, $P(\Phi)$ be the weight lattice. Fix an order on Φ, and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative and fundamental roots respectively. Our numbering of the fundamental roots follows that of [Bu1] (see Table 1). By $\overline{\omega}_1, \dots, \overline{\omega}_l$ one denotes the corresponding fundamental weights.

Let $W = W(\Phi)$ be the Weyl group of the root system Φ, i.e. the group generated by the set of fundamental reflections $w_{\alpha_1}, \dots, w_{\alpha_l}$. For brevity sake we write $s_i = w_{\alpha_i}$. Denote the set $\{s_i,\ 1 \le i \le l\}$, of fundamental reflections by S. A conjugate t of a fundamental reflection, $t = w w_{\alpha_i} w^{-1}$, is called a reflection and the set of all reflections will be denoted by T. Let l be the length function on W, i.e. $l(w)$ is the length of the shortest expression of $w \in$ W in terms of the fundamental reflections.

There are two partial orders on the group W called the *strong* and the *weak Bruhat orders*. Namely, for $u, v \in W$ we write $u \le v$ and say that u precedes v in the strong Bruhat order (usually called simply the Bruhat order), if there exist