Tame Minimal Non-Polynomial Growth Strongly Simply Connected Algebras

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Abstract

A finite dimensional basic connected algebra A over an algebraically closed field K is called strongly simply connected if every full convex subcategory of A is simply connected. We give a complete classification of tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations.

Throughout this article, K will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional K–algebra, which we shall assume to be basic. An algebra A can be written as a bound quiver algebra $A \simeq KQ/I$ where $Q = Q_A$ is the quiver of A and I is an admissible ideal in the path algebra KQ of Q. Equivalently, every algebra $A = KQ/I$ will equally be considered as a K–category of which the object class is the set of vertices of Q, and the set of morphisms from x to y is the vector space $KQ(x, y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x, y) = I \cap KQ(x, y)$. We shall call an algebra A triangular whenever its quiver Q_A has no oriented cycles. Following [AS] a triangular algebra A is called simply connected if, for any presentation $A \simeq KQ/I$ of A as a bound quiver algebra, the fundamental group $\Pi_1(Q, I)$ of (Q, I) is trivial. Further, A is called strongly simply connected if every full convex subcategory of A is simply connected [S1]. It was shown in [S1] that a triangular algebra A is strongly simply connected if and only if the first Hochschild cohomology $H^1(C, C)$ of any full convex subcategory C of A vanishes. This is also equivalent to the fact that every full convex subcategory C of A has the separation property (in the sense of [BLS].) Recall also that an algebra A is called tame, if for any dimension d, there is a finite number of $K[X]$–A–bimodules M_i which are finitely generated and free as left $K[X]$–modules and satisfy the following condition: all but a finite number of isomorphism classes of indecomposable right A–modules of dimension d are of the form $K[X]/(X-\lambda) \otimes_{K[X]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of bimodules M_i satisfying the above condition. Then A is called of polynomial growth if there is a natural number m such that $\mu_A(d) \leq d^m$ for all $d \geq 1$.

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It is shown in [S2],[S3], that a strongly simply connected algebra A is of polynomial growth if and only if A does not contain a convex subcategory which is hypercritical or pg-critical. The hypercritical algebras (being the preprojective tilts of minimal wild hereditary tree algebras) have been classified by quivers and relations in [U] (see also [L],[W]). The definition of pg-critical algebras presented in [S2] is in some general terms (see section 1), and does not allow to apply the above criterion in practice. Note that it follows from the above criterion that the class of strongly simply connected pg-critical algebras coincides with the class of tame strongly simply connected algebras which are minimal not of polynomial growth (they themselves are not of polynomial growth but every proper convex subcategory is of polynomial growth).

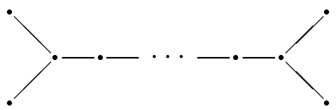
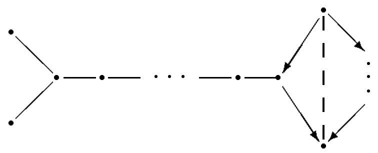
The aim of this paper is to establish a handy description of the tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations. In the course of our proofs we also obtain a description of tame minimal non-polynomial growth incidence algebras of finite partially ordered sets.

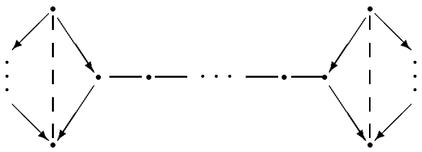
Throughout the paper, by an A–module always is meant a finitely generated right A–module, and we shall denote their module category by mod$A$. For each vertex i of Q_A we denote by $S_A(i)$ the corresponding simple A–module, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of $S_A(i)$. We shall denote by Γ_A the Auslander-Reiten quiver of A and by $\tau_A = \mathrm{DTr}$ and $\tau_A^- = \mathrm{TrD}$ the Auslander-Reiten translations. Finally, we shall call an A-module M thin, if the entries of its dimension vector $\underline{\dim} M = (\dim_K \mathrm{Hom}_A(P_A(i), M))_{i \text{ vertex in } Q_A}$ all are less or equal to 1. For a basic background from representation theory we refer to [Ri].

1 The main results

We shall present a complete description of tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations. From [S2] we know, that this class of algebras coincides with the class of strongly simply connected pg-critical algebras. We first recall the definition of pg-critical algebras. For an algebra $A = KQ/I$, generators of I are usually called relations. In our quivers, a dashed line indicates a relation being the sum of all paths from the starting point to the end point.

We know from [B], [HV] that there are only four families of critical algebras of type $\tilde{\mathbf{D}}_n, n \geq 4$, given by the following quivers and relations:

(1)  (2) 

(3)  (4)