On the convergence of asynchronous iteration methods for nonlinear paracontractions and consistent linear systems

Matthias Pott*

Abstract

In [6] the authors introduced the concept of paracontracting operators for fixed point problems and their solution with asynchronous iteration methods. We will extend their results essentially with respect to the criterion of contraction for the *pool* of operators, from which a common fixed point is searched and a more general kind of asynchronous iterations, which we will call confluent. As an application of this theory, we will consider asynchronous iteration methods for consistent linear systems $Ax = b$, where A is a singular M-matrix. In [7] Lubachevski and Mitra investigated asynchronous iteration methods for the Perron-vector problem, e.g. determining a positive solution of $Tx = \rho(T)\,x$, where T is an irreducible nonnegative matrix and its spectral radius is known. We will show that we can extend their results by using the developed theory of paracontractions and confluence to the reducible-affine case which arises, if we have those kind of linear systems.

Key words: Asynchronous iterations, paracontracting operators, nonnegative matrices, singular M-systems, Frobenius vectors.

AMS(MOS) subject classifications. 65F10, 65F15, 65J15.

1 Introduction

In [6] Elsner, Koltracht and Neumann studied the convergence of the sequential iteration

$$x(j+1) := G^{k(j)}(x(j)) \qquad j = 0, 1, \ldots \tag{1}$$

and the asynchronous iteration

$$x(j+1) := \alpha_j\, x(j) \;+\; (1-\alpha_j)\, G^{k(j)}(x(s(j))) \qquad j = 0, 1, \ldots\,, \tag{2}$$

*Research supported by Sonderforschungsbereich 343: „Diskrete Strukturen in der Mathematik“, Universität Bielefeld. Postfach 100131, 33501 Bielefeld.

where in both cases the $G^{k(j)}$'s were chosen from a finite *pool* of real operators $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$, here $\mathbb{K}$ is any set. This pool was assumed to be paracontracting, that is: All the operators $G : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathcal{G}$ are continuous, and for any fixed point ξ of an operator G of this pool, and for any $x \in \mathbb{R}^n$ either

$$\| G(x) - \xi \| < \| x - \xi \| \tag{3}$$

or x itself is a fixed point of G. The α_j's in (2) are chosen from a finite set of numbers from $[0,1)$, with the restriction that $\alpha_j = 0$ is only *permitted,* if $s(j) = j$ ($s(j)_{\,j=0,1,\ldots}$ is a sequence from $\mathbb{N}_0$, with $s(j) \le j \ \forall\, j \in \mathbb{N}_0$, while $k(j)_{\,j=0,1,\ldots}$ is a sequence from $\mathbb{K}$). The underlying model of parallelism in (2) is as follows: Assume we have K processors $\pi_1, \ldots, \pi_K$. At iteration step $s(j)$ processor $\pi_{k(j)}$ retrieves the global approximation $x(s(j))$, which resides in some shared memory, and computes a local iteration $G^{k(j)}(x(s(j)))$. At the *time* j, when this computation is done, the global iteration in the shared memory is updated by a convex combination as in (2). The classical asynchronous iteration scheme was introduced for linear systems by Chazan and Miranker in [4] and in a variant as an asynchronous *multisplitting*-method in [3]. In [1] this concept was extended to the nonlinear case. This scheme was introduced for only one operator $G : \mathbb{R}^n \to \mathbb{R}^n$:

$$x_i(j+1) := \begin{cases} x_i(j) & \text{, if } \ i = p_j \\ G_i\left(\begin{pmatrix} x_1(s^1(j)) \\ \vdots \\ x_n(s^n(j)) \end{pmatrix} \right) & \text{, if } \ i \ne p_j \end{cases} \qquad j = 0, 1, \ldots, \tag{4}$$

where again $s_i(j)_{\,j=0,1,\ldots}$, $i = 1, \ldots, n$ are sequences from $\mathbb{N}_0$, with $s_i(j) \le j \ \forall\, i, j$, and $p_{j\ j=0,1,\ldots}$ is a sequence from $\{1, \ldots, n\}$. The concept of parallelism here is substantially the same as above, but here the different processors update single or (as in the original, but after renumbering equivalent definition in [1]) groups of components of the system. Criteria of contraction were introduced in [1] and [10]. If the sequences $s^i(j)_{\,j=0,1,\ldots}$ and $p_{j\ j=0,1,\ldots}$ fulfil expectable conditions, then these criteria are necessary and sufficient for general convergence in the linear case, if the spectral radius of the Jacobian of G at its unique fixed point ξ satisfies $\rho(|G'(\xi)|) \ne 1$.
In the second section we will define the asynchronous iteration scheme, which we shall study. The convex combination in (2) is to guarantee something like a coupling in the iteration scheme. For example, let G^1 be an operator with fixed point ξ, and G^2 be one with fixed point $\eta (\ne \xi)$, and let both vectors arise as iterations in (2), then, if the α_j's were zero, without the restriction described above, it would be easy to construct divergent sequences, even if there is a common fixed point of the pool $\{G^1, G^2\}$, namely some, which alternate between η and ξ. If there is only one operator, which has two fixed points, the same phenomenon could arise, and the iteration process would not converge. We will formulate another concept of coupling, to avoid the convex combination in (2). Asynchronous iterations, which fulfil these coupling-postulates, will be called confluent.
In the third section we will collect some criteria of contraction for the pool $\mathcal{G}$, which