SETS OF INTEGERS AND QUASI–INTEGERS WITH PAIRWISE COMMON DIVISOR

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0. Introduction

Consider the set $\mathbb{N}_s(n) = \left\{ u \in \mathbb{N} : \left(u, \prod_{i=1}^{s-1} p_i \right) = 1 \right\} \cap \langle 1, n \rangle$ of positive integers between 1 and n , which are not divisible by the first $s-1$ primes $p_1, \dots, p_{s-1}$.

Erdös introduced in [4] (and also in [5], [6], [7], [9]) the quantity $f(n,k,s)$ as the largest integer ρ for which an $A \subset \mathbb{N}_s(n)$, $|A| = \rho$, exists with no $k+1$ numbers being coprimes. Certainly the set

$$\mathbb{E}(n,k,s) = \{ u \in \mathbb{N}_s(n) : u = p_{s+i} v \text{ for some } i = 0, 1, \dots, k-1 \} \tag{1}$$

does not have $k+1$ coprimes.

Conjecture 1 (Erdös [4]): $f(n,k,1) = |\mathbb{E}(n,k,1)|$ for all $n, k \in \mathbb{N}$ was disproved in [1].

This disproves of course also the General Conjecture (Erdös [7]): for all $n, k, s \in \mathbb{N}$

$$f(n,k,s) = |\mathbb{E}(n,k,s)|. \tag{2}$$

However, in [2] we proved (2) for every k, s and (relative to k, s) large n .

In the present paper we are concerned with the case $k = 1$, which in [1] and [2] we called

Conjecture 2: $f(n,1,s) = |\mathbb{E}(n,1,s)|$ for all $n, s \in \mathbb{N}$.

Erdös mentioned in [7] that he did not even succeed in settling this special case of the General Conjecture.

Whereas in [1] we proved this by a completely different approach for $n \geq (p_{s+1} - p_s)^{-1} \prod_{i=1}^{s+1} p_i$, we establish it now for all n (Theorem 2).

We generalize and analyze Conjecture 2 first for quasi–primes in order to understand how the validity of Conjecture 2 depends on the distribution of the quasi–primes and primes. Our main result is a simply structured sufficient condition on this distribution (Theorem 1). Using sharp estimates on the prime number distribution by Rosser and Schoenfeld [14] we show that this condition holds for $\mathbb{Q} = \{p_s, p_{s+1}, \dots\}$, $s \geq 1$, as set of quasi–primes and thus Theorem 2 follows.