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THE CANONICAL CLASS AND THE C^∞-PROPERTIES OF KÄHLER SURFACES.

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A compact complex surface X with non negative Kodaira dimension, has a unique minimal model $X_{\min}$. The pullback of the canonical line bundle minimal model $\omega_{\min}$ is in some ways the most basic birational invariant of the surface, if only because it is the polarisation $\mathcal{O}(1)$ of the canonical model $\mathrm{Proj}(\oplus H^0(nK))$. Recently, Kronheimer, Mrowka and Tian, Yau proved that the cohomology class, $K_{\min} = c_1(\omega_{\min}) \in H^2(X, \mathbb{Z})$ is invariant under oriented diffeomorphism up to sign for minimal surfaces of general type with $p_g > 0$ [Ste]. While completing this manuscript, Friedman and Morgan posted a proof for the case $p_g = 0$ [FM3]. In the case of elliptic surfaces it was already known to be true by the joint effort of many people, as it is a direct consequence of the invariance of the multiplicities of the elliptic fibration.

The proof is based on fundamental work of Witten and Seiberg [Wit], who introduced a new set of non linear equations, the monopole equations, which allow to define new differentiable invariants, similar in spirit to the Donaldson invariants, but much easier to handle. During the Stillwater conference in November 1994, Stefan Bauer and I learned about these new invariants and the invariance of $K_{\min}$, and we decided to run a seminar on Seiberg Witten theory in Bielefeld. I started to think about a proof for the invariance of $K_{\min}$ using the SW invariants along the lines of [Bru]. A little to my surprise, it worked out beautifully. The monopole equations which define the SW classes, once specialised to the Kähler case, give all the necessary information, without the necessity to prove a full Thom conjecture type of result.

In fact it even turned out to be possible to deal with the case $p_g = 0$ and the elliptic surfaces simultaneously and almost uniformly. There is no need to use any result from "classical " Donaldson theory either. From the point of view of classification of surfaces, it is very satisfactory that the various levels of nefness of $K_{\min}$ (nef and big, nef but not big, torsion) is what makes the proof work for Kodaira dimension $\kappa \geq 0$, what makes it fail for the rational and ruled case, and what makes for the difference in the different Kodaira dimensions. If $p_g = 0$ the higher plurigenera, and in particular the 2-canonical system plays a direct role.

While proving the invariance of the canonical class, we have to prove the invariance of (-1)-curves as well. This leads directly to the differentiable characterisation corollary 3 of rational and ruled surfaces which are characterised algebraically by the existence of a smooth rational curve l with $l^2 \geq 0$ [BPV, Prop. V.4.3]. The

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invariance of the Kodaira dimension (Van de Ven conjecture) and the invariance of the plurigenera for surfaces of general type is then almost an afterthought. The Van de Ven Conjecture had already been solved with Donaldson theory ([FM2] for all surfaces but rational surfaces and surfaces of general type with $p_g = 0$, and Friedman Qin [FQ] and Pidstrigatch [P-T],[Pi2] for the remaining cases, see also [OT1] for an easy proof of the remaining case with Seiberg Witten theory).

Theorem 1. *If X is a Kähler surface of non negative Kodaira dimension then*

1. *$K_{\min}$ is determined by the underlying oriented manifold up to sign,*
2. *every (-1)-sphere is $\mathbb{Z}$-homologous to a (-1)-curve up to sign.*

Corollary 2. *If a Kähler surface X has non negative Kodaira dimension then every smooth sphere S with $S^2 \geq 0$ is $\mathbb{Z}$-homologous to 0,*

Corollary 3. *A Kähler surface is rational or ruled if and only if it contains a smooth sphere $S \neq 0 \in H^2(X, \mathbb{Z})$ with $S^2 \geq 0$.*

Corollary 4. *The Kodaira dimension of a Kähler surface is determined by the underlying differentiable manifold.*

To deal with the case $p_g = 0$, we encounter higher dimensional moduli spaces and more to the point, moduli spaces that have larger than virtual dimension. However, following Pidstrigatch and Tyurin, we will identify the multiplicity of a Seiberg Witten class as a localised Euler class of an infinite rank bundle with a Fredholm section, and the oversized moduli spaces will cause no problem at all. In addition without much extra work, the computation will give us the Seiberg Witten multiplicities of all Kähler elliptic surfaces. It demonstrates my belief that Seiberg Witten theory for surfaces is completely computable. An elegant argument of Stefan Bauer, then gives yet another proof that for elliptic surfaces with finite cyclic fundamental group the multiplicities of the elliptic fibration are determined by the underlying oriented manifold. The oriented homotopy type determines the multiplicities for other elliptic surfaces [FM2, Theorem S.7]. By the first two chapters of [FM2] (now probably the most difficult part of the story) this implies

Theorem 5. *Let $X \to C$ be an elliptic Kähler surface. Then the multiplicities of the elliptic fibration are determined by the underlying oriented smooth manifold. In particular for Kähler elliptic surfaces deformation equivalent, and oriented diffeomorphic are the same notions.*

This theorem has been well established with Donaldson theory by the work of Bauer, Kronheimer, Fintushel, Friedman, Morgan, Mrowka, O'Grady and Stern.

Corollary 6. *The plurigenera of a Kähler surface are determined by the underlying oriented manifold.*

Let me remark that it seems to be known that in the non Kähler case, with the exception of the equivalence of deformation and diffeomorphism equivalence of non Kähler elliptic surfaces, (where there can be a two to one discrepancy) all the previous statements are true as well, but seemingly for "classical" reasons like the homotopy type.

Inspired by results in the preprint of Friedman and Morgan I realised how the results in this article give an easy proof of

Corollary 7. *No Kähler surface of non negative Kodaira dimension admits a metric of positive scalar curvature*