Bicombings of triangle buildings

Gennady A. Noskov

April 4, 1995

Abstract

For an arbitrary triangle building Δ we define some "rational" bicombing, which satisfies the "k-fellow traveller property" and which is invariant under "type rotating" automorphisms of Δ. In particular we obtain a purely geometric proof of biautomacity of a finitely generated group,acting simply transitively on the vertices of Δ.

Introduction

The idea of bicombing, introduced by W.Thurston, became quite popular in combinatorial group theory, see, for example [AB], [ECHLPT], [GS], [NS]. A bicombing of a space is a choice of a family of paths in X between each (or almost each) pair of points of X. The idea is that bicombings with rather nice dependence of the paths on varying end points generalize the geometry of geodesics in a simply connected manifold of nonpositive curvature. This approach permits [AB] *"... to transfer as much information as possible about the asymptotic geometry of nice spaces, such as nonpositively curved manifolds, to related spaces whose lack of local structure precludes local-to-global analysis"*.

Every affine building, being a CAT(0)- space admits a geodesic bicombing. In this paper we are interested in bicombings which are "sufficiently rational", having in mind to apply this to constructing biautomatic structures on groups. The following definition of rationality seems to be good enough: a bicombing $\mathcal{C}$ on an affine building Δ is *rational* if

1)For every pair of vertices of Δ there is a path $\gamma \in \mathcal{C}$ joining these points,

2) Every $\gamma \in \mathcal{C}$ lies in the 1-dimensional skeleton of Δ,

3) There exists a natural number n, such that every $\gamma \in \mathcal{C}$ consists of at most n geodesics,

4) $\mathcal{C}$ satisfies the "fellow traveller property" (see section 3 for definition).

We believe that a rational bicombing exists on every affine building. In this paper we construct such a bicombing on every triangle building ($\tilde{A}_2$–building in Tits' classification). This permits us to obtain a purely geometric proof of the biautomacity of a finitely generated group acting simply transitively on the vertices of a triangle building, see [CS].

I am grateful to the SFB 343, University of Bielefeld for their hospitality while I carried out much of this work. I would like to thank Herbert Abels for his kind invitation, interest and support.

The starting point was the paper [CMSZ] and I am grateful to Sasha Borovik for bringing that paper to my attention. I gratefully acknowledge discussions with R. Bieri, V. Gerasimov and W. D. Neumann.

I would like to thank Herbert Abels for his kind invitation, interest and support. I gratefully acknowledge the suggestions of Herbert Abels and John Klein, improving the text.

1 Triangle buildings

1.1 Definitions

We adopt the following direct definition of a triangle building which can be easily extracted from [Bro]. Let Σ_0 be the abstract simplicial complex associated with the Euclidean plane, tiled by equalaterial triangles. We call a simplicial complex Δ a triangle building if it can expressed as the union of a family of subcomplexes Σ, called apartments,satisfying the following conditions:

B0) Each apartment Σ is isomorphic to Σ_0,

B1) Any two simplices of Δ are contained in an apartment,

B2) Given two apartments Σ, Σ' with a common 2-simplex (=chamber), there is an isomorphism $\Sigma \xrightarrow{\approx} \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.

We prefer to consider the geometric realization $|\Delta|$ of Δ rather than Δ itself. One can easily show that there is a canonical bijection between $|\Sigma_0|$ and the Euclidean plane, which Σ_0 triangulates. Thus Σ_0 carries a metric, which we normalize, agreeing that all 1-simplices(edges) have length 1. If we do the same for each apartment Σ, then we see that the simplicial isomorphism $\Sigma \xrightarrow{\approx} \Sigma'$ in the definition of Δ extends uniquely to an isometry $\Sigma \xrightarrow{\approx} \Sigma'$.

It will be convenient to apply to $|\Delta|$ the same terminology as to Δ itself.In particular, we will refer to $|\Delta|$ as a building and to the subsets $E = |\Sigma|$ as apartments. Moreover, we will use the notation Δ for $|\Delta|$.

So each apartment E of Δ is a Euclidean space with a metric $|x-y|_E, x, y \in E$. Moreover, the isomorphisms between apartments given by the building axiom (B2) are isometries. The metrics $|x-y|_E$ can be pieced together to make the entire building Δ a metric space. The resulting metric will be denoted by $x, y \mapsto |x-y|$. It is known (see [Bro],Thm. VI.3) that the metric space Δ is complete. Besides, for any $x, y \in \Delta$ the line segment $[x, y]$ joining them in any apartment E, the line segment $[x, y]$ is independent of the choice of E and can be characterized by

$[x, y] = \{z \in X; |x-y| = |x-z| + |z-y|\}$

Besides, $[x, y]$ is *geodesic*, that is the shortest path joining x and y and there is no other geodesic joining x and y.

Let us call a subset $M \subset \Delta$ *Euclidean* if it is contained in some apartment. The subset of Δ is called *convex* if it contains the geodesic connecting any two of its