FIRST ORDER PERTURBATIONS OF DIRICHLET OPERATORS: EXISTENCE AND UNIQUENESS

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Abstract. We study perturbations of type $B \cdot \nabla$ of Dirichlet operators $(L^0, D(L^0))$ associated with Dirichlet forms of type $\mathcal{E}^0(u,v) = 1/2 \int \langle \nabla u, \nabla v\rangle_H \, d\mu$ on $L^2(E,\mu)$ where E is a finite or infinite dimensional Banach space E. Here H denotes a Hilbert space densely and continuously embedded in E. Assuming quasi–regularity of $(\mathcal{E}^0, D(\mathcal{E}^0))$ we show that there always exists a closed extension of $Lu := L^0 u + \langle B, \nabla u\rangle_H$ that generates a sub–Markovian C_0–semigroup of contractions on $L^2(E,\mu)$ (resp. $L^1(E,\mu)$), if $B \in L^2(E; H, \mu)$ and $\int \langle B, \nabla u\rangle_H \, d\mu \leq 0, \ u \geq 0$. If D is an appropriate core for $(L^0, D(L^0))$ we show that there is only one closed extension of (L, D) in $L^1(E,\mu)$ generating a strongly continuous semigroup. In particular we apply our results to operators of type $\Delta_H + B \cdot \nabla$, where Δ_H denotes the Gross–Laplacian on an abstract Wiener space (E, H, γ) and $B = -\mathrm{id}_E + v$, where v takes values in the Cameron–Martin space H.

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0. Introduction

In this paper we are concerned with the problem whether certain (non–symmetric) first order perturbations of Dirichlet operators on finite or in particular infinite dimensional state spaces have extensions which generate sub–Markovian C_0–semigroups and, if so, whether there is only one such extension.

In order to explain our results more precisely we first consider the following special finite dimensional example: Denote the closure of

$$1/2 \int \langle \nabla u, \nabla v \rangle \varphi^2 dx;\ u, v \in C_0^\infty(\mathbb{R}^d)$$

in $L^2(\mathbb{R}^d, \varphi^2 dx)$ by $(\mathcal{E}^0, D(\mathcal{E}^0))$, where $\varphi \in L^2(\mathbb{R}^d, dx)$ with $|\nabla\varphi| \in L^2_{\mathrm{loc}}(\mathbb{R}^d, dx)$. Denote the associated generator by $(L^0, D(L^0))$ and let $T_t^0 = e^{tL^0}$, $t \geq 0$, be the corresponding semigroup. Let B be a measurable vector field on $\mathbb{R}^d$ and assume that $|B| \in L^2(\mathbb{R}^d, \varphi^2 dx)$. We can then define the linear operator

$$Lu := L^0 u + \langle B, \nabla u \rangle;\ u, v \in C_0^\infty(\mathbb{R}^d)\,.$$

Question 1: Is there a linear operator $(L, D(L))$ on $L^2(\mathbb{R}^d, \varphi^2 dx)$ (resp. $L^1(\mathbb{R}^d, \varphi^2 dx)$) extending $(L, C_0^\infty(\mathbb{R}^d))$ such that $(L, D(L))$ generates a C_0–semigroup $(T_t)_{t\geq 0}$ which is sub–Markovian (i.e. $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$ for all $t \geq 0$) ?
Question 2: If the answer to Question 1 is yes, is $(L, D(L))$ having these properties unique ?

Assuming $\int |x|^2 \varphi^2(x)\, dx < \infty$ the answer to Question 1 is positive under the following condition

$$\int \langle B, \nabla u \rangle\, \varphi^2 dx \leq 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d),\ u \geq 0\,. \tag{0.1}$$

The proof relies on the application of a method due to E. A. Carlen (cf. [Ca]) which he used in the construction of conservative diffusions and which is partly generalized to our situation. We approximate B by bounded vector fields B^n, $n \geq 1$, such that $|B - B^n| \to 0$ in $L^2(\mathbb{R}^d, \varphi^2 dx)$. For every n we can define the operator $L^n u := L^0 u + \langle B^n, \nabla u \rangle$, $u \in D(L^0)$, that obviously generates a strongly continuous sub–Markovian semigroup $(T_t^n)_{n\geq 1}$ on $L^2(\mathbb{R}^d, \varphi^2 dx)$. By (0.1) it is then easy to see that $(T_t^n u)_{t\geq 0}$ is an $L^2(\mathbb{R}^d, \varphi^2 dx)$–Cauchy sequence for bounded u and that the limit $T_t u$ can be extended to a sub–Markovian contraction operator on $L^2(\mathbb{R}^d, \varphi^2 dx)$ as well as on $L^1(\mathbb{R}^d, \varphi^2 dx)$. However, in order to prove that the family $(T_t)_{t\geq 0}$ is a strongly continuous semigroup whose generator indeed extends $(L, C_0^\infty(\mathbb{R}^d))$ we have to use probabilistic tools, i.e. diffusions $\mathbf{M}^0 = (\Omega, \mathcal{M}^0, (X_t)_{t\geq 0}, (P_x^0)_{x\in\mathbb{R}^d})$ and $\mathbf{M}^n = (\Omega, \mathcal{M}^n, (X_t)_{t\geq 0}, (P_x^n)_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$ with transition probabilities given by $(T_t^0)_{t\geq 0}$ resp. $(T_t^n)_{t\geq 0}$. The family $(T_t)_{t\geq 0}$ can then be identified as the transition semigroup of some diffusion process $\mathbf{M}^B = (\Omega, \mathcal{M}^B, (X_t)_{t\geq 0}, (Q_x)_{x\in\mathbb{R}^d})$ obtained as the Girsanov–transform of $\mathbf{M}^0$.

We would like to emphasize that the role of the Laplacian in the above discussion can be replaced by a more general second order strongly elliptic operator.