Regularity of invariant measures: the case of non-constant diffusion part

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October 30, 1994

Abstract

We prove regularity (i.e., smoothness) of measures μ on R^d satisfying the equation $L^*\mu = 0$ where L is an operator of type $Lu = tr(Au'') + B \cdot \nabla u$. Here A is a Lipschitz continuous, uniformly elliptic matrix-valued map and B is merely μ-square integrable. We also treat a class of corresponding infinite dimensional cases where $\mathbb{R}^d$ is replaced by a locally convex topological vector space X. In this cases μ is absolutely continuous w.r.t. a Gaussian measure on X and the square root of the Radon-Nikodym density belongs to the Malliavin test function space $\mathbb{D}^{2,1}$.

AMS Subject Classification (1991)
Primary: 35B65, 60H10
Secondary: 35R15, 60H15, 47D06, 28C20

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1 Introduction and the main result in finite dimensions

The purpose of this paper is to prove regularity (i.e., smoothness) of measures μ satisfying certain elliptic equations $L^*\mu = 0$ in finite and infinite dimensions where L is a second order operator with non-constant diffusion part and possibly very singular drift. Let us first state our main result in finite dimensions.

Consider the operator

$$\begin{aligned} Lu &= tr(Au'') + (\nabla u, B)\ , \ \ u \in C_b^\infty(\mathbb{R}^d) \\ &= a^{ij}\partial_i\partial_j u + B^i\partial_i u\ . \end{aligned} \tag{1.1}$$

Here we use the summation convention, ∂_i means derivative w.r.t. the i-th coordinate, $(\ ,\ )$ denotes the Euclidean inner product on R^d, $B := (B^i) : \mathbb{R}^d \to \mathbb{R}^d$ is Borel-measurable, and $A := (a^{ij}) : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ is a matrix-valued map which is bounded, uniformly Lipschitz continuous and uniformly strictly elliptic with constant α. Without loss of generality we may assume that $A(x)$ is symmetric for all $x \in \mathbb{R}^d$. Let μ be a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$. We write

$$L^*\mu = 0 \tag{1.2}$$

if

$$|B| \in L^2(\mu)\ \ (:= L^2(\mathbb{R}^d;\mu))\ . \tag{1.3}$$

and

$$\int Lu\, d\mu = 0 \quad \forall u \in C_b^\infty(\mathbb{R}^d) \tag{1.4}$$

Let dx denote Lebesgue measure on $\mathbb{R}^d$.

Theorem 1.1 *Let μ be a pobability measure on $\mathcal{B}(\mathbb{R}^d)$ such that $|B| \in L^2(\mu)$ and $L^*\mu = 0$. Then*

(i) $\mu = p\,dx$ where $p = \varphi^2$ with $\varphi \in H^{1,2}(\mathbb{R}^d;dx)$ (i.e., the Sobolev space of order 1 in $L^2(\mathrm{R}^d;dx)$). In particular, $p \in L^{d/(d-1)}(\mathbb{R}^d;dx)$.

(ii) The following holds for p, φ :

$$\frac{1}{4}\int\left|\frac{\nabla p}{p}\right|^2 p\,dx = \int |\nabla\varphi|^2 dx \le \frac{1}{4\alpha^2}\int |B+D|^2 d\mu\ , \tag{1.5}$$

where $D := (D^j) := (\partial_i a^{ij})$, $|\ \ |$ denotes Euclidean norm (and the derivatives are meant in the sense of distributions).

(iii) $\frac{\nabla p}{p}$ coincides $\mu - a.e.$ with the orthogonal projection of the vector field $A^{-1}(B-D)$ onto the closure of $\{\nabla u|\ u \in C_0^\infty(\mathbb{R}^d)\}$ in $L^2(\mu, \mathbb{R}^d)$ $(:= L^2(\mathbb{R}^d \to \mathbb{R}^d;\mu))$ equipped with the inner product $(\ ,\ )_2$ defined by

$$(F,G)_2 := \int (AF, G)\, d\mu\ .$$