Brauer groups of local elliptic and hyperelliptic curves and central division algebras over their function fields

by
Vyacheslav Yanchevskii
and
Gennady Margolin

Abstract. We compute the torsion part of the unramified Brauer group of the function field of any elliptic curve defined over a local field of characteristc zero. For the case of a non dyadic ground field we prove that all elements of the 2-torsion part can be represented by quaternion algebras and describe them in an explicit form. We obtain a similar result for hyperelliptic curves with good reduction such that their defining equation $y^2 = f(x)$ satisfies the condition $\deg f(x) \not\equiv 0 (\mathrm{mod}\ 4)$.

Contents

1 Introduction

Let $K|k$ be a function field, i.e. assume that the field K is finitely generated, as a field, over the field k and let $Br_{nr}(K|k)$ be the unramified Brauer group of K (with respect to k). Let X be a smooth projective connected variety over field k of characteristic zero and $K = k(X)$ be its function field. It is known (see e.g. [6]) that there is a natural inclusion of $Br\ X$ in $Br(K)$ ($Br\ X$ denotes the Brauer group of X [5]) and this inclusion identifies $Br\ X$ with uramified Brauer group $Br_{nr}(K|k)$. (Below we shall write $Br\ X$ instead of $Br_{nr}(K|k)$ keeping in mind this identification).

The group $Br\ X$ is of great importance for many problems (it is enough to mention the rationality problem of varieties, the problem of Brauer-Manin obstructions to various Hasse principles and finally the classical problem of description of finite-dimensional division algebras over function fields). (see [6],[20], [21].)

In this paper we shall be interested in the problem of the description of $Br\ X$ for smooth geometrically connected projective curves over local fields, more precisely, for finite extention k of $\boldsymbol{Q}_p$. The classical class field theory gives us a complete description of the Brauer group $Br(k)$ of k (and also of finite dimensional central division algebras over k). But if $k(X)$ is a function field of a smooth geometrically connected projective variety over k the situation is much more complicated. In spite of many results obtained in connection with the developing of multidimensional class field theory ([18],[19]), unfortunately, we have not such a complete description as in the classical situation even in case of curves X (a "complete" description here means a presentation of each element of $Br(k(X))$ by the corresponding central division algebra over $k(X)$, in other words, it means a description of all central division algebras over $k(X)$). The Tradition of local-to-global principles suggest to investigate the local before the global situation . In case of curves X over k it means to investigate the groups $Br(k_v(X_v))$ for all v, where k_v is a completion of k with respect to a valuation v of k, and $X_v = X \times_k k_v$. (By Tsen's theorem $Br(k_v(X_v))$ is trivial in case of a complex v and its structure is well-known for real v according to [1], [2], [3] and [17]).

Thus the main unknown case in this situation is the case of a non-archimedian v, so below by a local field we shall understand a finite extension of the field of p-adic numbers $\boldsymbol{Q}_p$. Let now X be a smooth geometrically connected projective curve over local field k. Then it follows from [4], that there is an exact sequence

$$0 \longrightarrow Br\ X \longrightarrow Br(k(X)) \xrightarrow{\phi} \coprod_{P\in X} \chi(G_P) \xrightarrow{\psi} \chi(G) \longrightarrow 0,$$

where for a separable closure k_s of k, $G = Gal(k_s|k)$, $G_P = Gal((k_P)_s|k_P)$, k_P is the residue field of $P \in X$ and $\chi(G_P)$, $\chi(G)$ are the groups of continuous characters of G_P and G respectively (for definitions of ϕ and ψ see [4]). In view of this sequence the group $Im\ \phi$ is known, so the group of $Br\ X$ is of main interest for the description of $Br(k(X))$. The structure of $Br\ X$ is well-known in the case of curves of genus zero, but even in the case of elliptic curves the complete structure of $Br\ X$ is unknown. More generally, one can consider a smooth connected projective model X of an affine curve defined by an equation $y^2 = f(x)$, where $f(x)$ is a polynomial with coefficients in k without multiple roots and one can ask: what is known about the structure of $Br\ X$? One of the goals of our paper is to answer some questions related to the previous one. Since $Br\ X$ is an abelian periodic