SHADOWS AND ISOPERIMETRY UNDER THE SEQUENCE–SUBSEQUENCE RELATION

Rudolf Ahlswede and Ning Cai

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

1. Introduction and results

One of the basic results in extremal set theory was discovered in [1] and rediscovered in [2]: For a given number of k–element subsets of an n–set the shadow, that is, the set of $(k-1)$–element subsets contained in at least one of the specified k–element subsets, is minimal, if the k–element subsets are chosen as an initial segment in the squashed order (see [10]; called colex order in [11]), that is, a k–element subset A precedes a k–element subset B, if the largest element in $A \triangle B$ is in B. A closely related result was discovered in [3] and rediscovered in [5]: For a given number $u \in [0, 2^n]$ of arbitrary subsets of an n–set the "Hamming distance 1"–boundary is minimal for the initial segment of size u, also called in short " u–th initial segment", in the H–order (of [3]), that is, if one chooses all subsets of cardinality less than $n-k$ (k suitable) and all remaining subsets of cardinality $n-k$, whose complements are in the initial segment of the squashed order.

In this paper we consider sequences and subsequences rather than sets and subsets.

The basic objects are $\mathcal{X}^n = \prod^n \mathcal{X}$ for $\mathcal{X} = \{0,1\}$ and $n \in \mathbb{N}$, and operations of deletion $\bigtriangledown_i$, $\bigtriangledown$ and of insertion $\triangle_i$, $\triangle$. Here $\bigtriangledown_i$ (resp. $\triangle_i$) means that letter i $(i = 0,1)$ is deleted (resp. inserted) and $\bigtriangledown$ (resp. $\triangle$) means the deletion (resp. insertion) of any letter.

So for $A \subset \mathcal{X}^n$ we get the *down shadow*

$$\bigtriangledown A = \{x^{n-1} \in \mathcal{X}^{n-1} : x^{n-1} \text{ is subsequence of some } a^n \in A\} \tag{1.1}$$

and the *up shadow*

$$\triangle A = \{x^{n+1} \in \mathcal{X}^{n+1} : \text{ some subsequence of } x^{n+1} \text{ is in } A\}. \tag{1.2}$$

In other words, $\bigtriangledown A$ are all sequences of length $n-1$ obtained by omitting any letter in the sequences of A. Then $\bigtriangledown_i A$ are all those sequences obtained by omitting the letter i. Clearly,

$$\bigtriangledown A = \bigtriangledown_0 A \cup \bigtriangledown_1 A \tag{1.3}$$

and analogously

$$\triangle A = \triangle_0 A \cup \triangle_1 A. \tag{1.4}$$

We describe now our results.

A. Shadows for fixed level and specific letter.

The ℓ–th level is the set of sequences (or words)

$$\mathcal{X}^n_\ell = \left\{ x^n \in \mathcal{X}^n : \sum_{t=1}^n x_t = \ell \right\}. \tag{1.5}$$

We consider sets $B \subset \mathcal{X}^n_{n-k}$ of cardinality v, $0 \le v \le \binom{n}{k}$, and their shadows $\bigtriangledown_0 B$, $\triangle_1 B$ (the other shadows can be estimated similarily by symmetry).