THE GOODWILLIE TOWER OF THE IDENTITY IS A LOGARITHM

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ABSTRACT. We offer the point of view that the Goodwillie tower of the identity functor is a formal inverse to the standard filtration of stable homotopy in the same sense as $\ln(x)$ is inverse to $\exp(x)$.

0. INTRODUCTION

The *Goodwillie tower of the identity* (see [2, 3, 4] for a general reference on the Goodwillie calculus) is a tower of functors and natural transformations, which starts with stable homotopy and converges to unstable homotopy. The functors in this tower are characterized by certain universal properties, and according to [4] they should be thought of as the Taylor polynomials of the identity. The study of these polynomial approximations was initiated in [5] and continued in [1]. In particular, an explicit description of the fibers in the tower was derived there. However, the nature of the tower and the reason that it converges remained a bit mysterious. It appears that the Goodwillie tower is a natural, canonical object, and therefore one expects to have a basic understanding of it in terms of some fundamental object in mathematics. Our goal in this short, simple note is to show that there is a formal analogy between the Goodwillie tower of the identity and the Taylor series of the logarithm function and to suggest the point of view that the Goodwillie tower is an inverse to stable homotopy in the same way as logarithm is an inverse to exponential.

Let $D_n(X)$ denote the n-th fiber in the Goodwillie tower of the identity (the n-th differential). It was shown in [1] (see also [5] for a different approach) that $D_1(X) =$

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$\Omega^\infty\Sigma^\infty(X)$ and that for $n > 1$

$$D_n(X) \simeq \Omega^\infty \left(\mathrm{Map}_* \left(SK_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}\right)$$

where K_n is the unreduced suspension of the Folkman complex (essentially the geometric realization) of the category of partitions of the set $\underline{n} = \{1, \ldots, n\}$, and SK_n is the suspension of K_n. SK_n is a complex with a natural action of the symmetric group Σ_n. Furthermore, it is well-known that non-equivariantly

$$SK_n \simeq \bigvee_{i=1}^{(n-1)!} S^{n-1}.$$

Thus, the coefficient of the n-th differential of the identity looks like $K_n/\Sigma_n \approx \frac{(n-1)!}{n!} = \frac{1}{n}$, which reminds one of the Taylor series of the function $g(x) = \ln(1+x)$. We will see that this is not a coincidence, that the fact that the Goodwillie tower converges is a jazzed up version of the fact that the Taylor series of logarithm is inverse to the Taylor series of exponential. More precisely, our point of view is that stable homotopy is analogous to the function e^{x-1} rather than to a linear function, and the Goodwillie tower is an infinite product, rather than an infinite sum, namely it is analogous to the product

$$e^{x-1} \cdot e^{-\frac{(x-1)^2}{2}} \cdot e^{\frac{(x-1)^3}{3}} \cdots = e^{\ln(1+(x-1))} = x.$$

It is clear that in the context of the Goodwillie tower it is not good enough to think of the coefficient of the n-th term in the Taylor expansion of the function $\ln(1+x)$ as $\frac{1}{n}$. Rather, it is $\frac{\underline{(n-1)!}}{n!}$, where $\underline{(n-1)!}$ is a fancy $(n-1)!$, as given by the reduced homology, equivalently by the reduced Euler characteristic, of the partition lattice, considered as a Σ_n-representation. As a biproduct we will obtain an elementary proof of the well-known fact that the reduced Euler characteristic of K_n is $(-1)^n(n-1)!$. We will see that in our context the partition lattice is the "correct" way to think of the umbral coefficient in the Taylor tower of $\ln(1+x)$. We believe it to be true in other situations too. We emphasize, however, that the combinatorics in this paper is elementary and the corresponding topological results about the Goodwillie tower of the identity, as well as their proofs, are contained in [1]. It is their juxtaposition which we hope will be of some interest.

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1. Combinatorics versus homotopy theory

In the category of topological spaces the categorical product (cartesian product) and the categorical sum (disjoint union) satisfy the distributivity relation

$$X \times \left(A \coprod B\right) \cong (X \times A) \coprod (X \times B),$$