On the second real eigenvalue of nonnegative and Z–matrices *

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Abstract

We give bounds for the second real eigenvalue of nonnegative matrices and Z–matrices. Furthermore, we establish upper bounds for the maximal spectral radii of principal submatrices of nonnegative matrices. Using this bounds we prove that our inequality for the second real eigenvalue of the incidence matrix of a connected regular graph improves a well known bound for the second eigenvalue using Cheeger's inequality.

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1 Introduction

Let $\mathbb{R}^{n,n}$ be the set of $n \times n$ real valued matrices. Assume that $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and arrange the eigenvalues of A as follows

$$Re(\lambda_n(A)) \geq Re(\lambda_{n-1}(A)) \geq \ldots \geq Re(\lambda_1(A)).$$

Suppose that A is nonnegative ($A \geq 0$). Then $\lambda_n(A) = \rho(A)$ is the spectral radius of A. In many applications one needs to estimate the separation between $\rho(A)$ and $Re(\lambda_{n-1})$. See for example [F2] and the references therein. For symmetric nonnegative irreducible matrices the known inequalities are matrix variations of Cheeger's inequality [C]. For general nonnegative irreducible matrices a matrix version of Cheeger's inequality is given in [FG].

$$Re(\lambda_{n-1}(A)) \leq \rho(A) - \frac{1}{2}(\rho(A) - \max_{1\leq i\leq n} a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A)^2,$$

$$\epsilon_{\lfloor \frac{n}{2} \rfloor}(A) = \inf_{U\subset <n>, |U| \leq \lfloor \frac{n}{2} \rfloor} \frac{\sum_{i\in U, j\in <n>\setminus U} a_{ij}v_i u_j + a_{ji}v_j u_i}{\sum_{i\in U} 2(\rho(A) - a_{ii})v_i u_i},$$

where $u, v \in \mathbb{R}^{n,n}$ are the right and left positive eigenvectors corresponding to $\rho(A)$.

The aim of this paper is to give a different type of estimate on the real eigenvalue of A different from $\rho(A)$. This estimate is given in terms of spectral radii of principle submatrices of A. Namely, for $U \subset < n >$ let $A(U)$ be the principle submatrix of A whose rows and columns are in U. Set

$$\rho_m(A) = \max_{U\subset <n>, |U|=m} \rho(A(U)).$$

We then show that any real eigenvalue $\lambda, \lambda \neq \rho(A)$ satisfies $\lambda \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$. In the case that A is a symmetric nonnegative matrix, or more generally A is diagonally similar to a nonnegative symmetric matrix (in particular time reversible Markov chains), we have the inequality $\lambda_{n-1}(A) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$. For any nonnegative irreducible matrix we show

$$\rho_{\lfloor \frac{n}{2} \rfloor}(A) \leq \rho(A) - \frac{1}{2}(\rho(A) - \max_{1\leq i\leq n} a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A)^2. \tag{1.1}$$

It then follows that our inequality $\lambda_{n-1}(A) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$ for the incidence matrix A of a connected regular graph improves a well known bound for the second eigenvalue using Cheeger's inequality, e.g. [F2].