PAIRS OF SHORT ROOT SUBGROUPS IN A CHEVALLEY GROUP

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§ 1. Introduction

The purpose of this paper is to describe subgroups of a Chevalley group G of type B_l, C_l or F_4, generated by a pair of unipotent short root subgroups. In fact we do somewhat more: namely, we classify orbits of a Chevalley group acting by simultaneous conjugation on pairs of unipotent short root subgroups. In the sequel of this paper we consider analogous problems for the case of the Chevalley group of type G_2.

For unipotent long root subgroups such a description is very well known, see M.Aschbacher and G.M.Seitz [AS]. Namely, it turns out that any pair (X_1, X_2) of long root subgroups is simultaneously conjugate to a pair of elementary long roots subgroups (X_α, X_β). The orbits of G on pairs of root subgroups are essentially determined by the angle between α and β, so that generically there are five possible configurations, some of which may give two or three orbits, compare Theorem A below.

This result played an important role in understanding the irreducible subgroups generated by long root subgroups. Geometry of long root subgroups in Chevalley groups is now a well established field, see, for example, J.McLaughlin [M1], [M2], B.S.Stark [S1] – [S3], H.Pollatsek [P], A.Wagner [W], A.E.Zalesskii and V.N.Serezhkin [ZS], W.M.Kantor [K1], B.N.Cooperstein [C1] – [C4], R.Brown

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1991 *Mathematics Subject Classification.* 20G15, 20G40, 20D06.

Key words and phrases. Chevalley group, geometry of root subgroups, Bruhat decomposition, orbits of the conjugation action.

and S.P.Humphries [BH], Li Shang Zhi [L1] – [L3], F.G.Timmesfeld [T1] – [T5], M.W.Liebeck and G.M.Seitz [LS], H.Cuypers [Cu].

Irreducible subgroups of the classical groups generated by short root subgroups have been classified by B.S.Stark [S2] and Li Shang Zhi [L3]. Nevertheless the geometry of short root subgroups is far from being properly understood. To the best of our knowledge even for the classical cases the orbits of G on pairs of short root subgroups were not classified.

In [V1] – [V3] N.A.Vavilov calculated Bruhat decomposition of root elements. Using these results he classified the orbits of a Chevalley group on pairs of a long and a short root subgroups [V2]. In this case generically there are six possible configurations, compare Theorem B below. This is the starting point of our work.

In the present paper we make the next step and get a similar list for pairs of two short root subgroups. However our list is considerably longer, for odd characteristics there are 21 possible configurations (not all of them occur for each type), some of which split into infinitely many orbits, parametrised by a continuous parameter. This result is summarized in Theorem 1. For characteristic 2 short root subgroups behave exactly as the long root ones and the answer is much simpler, see Theorem 2.

One of the reasons why the answer is so much more complicated is that a short root subgroup (unlike a long root one) does not lie in the center of the unipotent radical U of a Borel subgroup, so that conjugation by elements of U leads to much more complicated configurations of roots. In fact, the mutual position of two long root subgroups depends only on two long roots and that of a pair consisting of a long and a short root subgroups on two or three roots (see [V2]). But the mutual position of two short root subgroups depends on two, three, four and in two cases on five roots.

The present paper is organized as follows. In § 2 we recall the known results about pairs of long – long (Theorem A) and long – short (Theorem B) subgroups and state our principal results about pairs of short – short subgroups (Theorems 1 and 2). The rest of the paper is devoted to the proofs of these results. Namely, in § 3 using the Bruhat decomposition of short root subgroups we reduce description of orbits to a combinatorial problem about roots. In § 4 for each type we draw a list of pairs, (X_1, X_2), which contains representatives of all orbits (Tables 5 – 7). In § 5 we identify the corresponding spans $X = \langle X_1, X_2 \rangle$ and study long and short root subgroups contained in X. Finally, in § 6 (for characteristic not 2) and § 7 (for characteristic 2) we prove that pairs corresponding to different configurations are not conjugate and determine non-conjugate pairs within the same configuration.

One can find further details related to these problems, as well as some applications and many additional references, in the surveys [C3], [K], [K2], [T4], [V4], [V5], [Z] and in the fundamental works of B.Cooperstein [C2], W.M.Kantor [K1], M.W.Liebeck and G.M.Seitz [LS] and F.Timmesfeld [T2], [T3], [T5]. In the sequel of this paper we hope to be able to imitate one of the main results of [C2] and to classify some important classes of subgroups generated by triples of short root