Co-Multiplication and Suspension

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1 Introduction

It has been known for a long time that an $(n-1)$-connected CW-complex of dimension $\leq 2n-1$ has the homotopy type of a suspension. In 1963 Bernstein and Hilton proved that an $(n-1)$-connected based CW-complex X of dimension $\leq 3n-3$ has the homotopy type of a suspension provided X admits a co-multiplication $X \to X \vee X$ with homotopy co-unit [1] (Bernstein and Hilton made the additional technical requirement that the homology groups of X are finitely generated). In 1970 Ganea extended this result: An $(n-1)$-connected co-H-space X of dimension $\leq 4n-5$, $n \geq 2$, has the homotopy type of a suspension ΣY if and only if it is homotopy co-associative. Moreover, the homotopy equivalence $\Sigma Y \to X$ is a homomorphism up to homotopy [5]. Bernstein-Hilton and Ganea conjectured that a suitable theory of co-A_n-spaces, dual to Stasheff's theory of A_n-H-spaces [12], would lead to extensions of their results. Indeed, in the A_n-space situation the following result holds [13], [16].

1.1 Proposition: Let X be an $(n-1)$-connected well-pointed A_{k-1}-space, $k \leq 3$, admitting a homotopy inverse, such that $\pi_i X = 0$ for $i > (k+1)n-4$. Then there is an A_{k-1}-homomorphisms $f : X \to \Omega_M Y$ which is a weak equivalence. Here $\Omega_M Y$ is the Moore loop space on Y with its natural monoid structure.

Recall that an A_k-homomorphism is a homomorphism up to homotopy satisfying coherence conditions up to $(k-1)$-dimensional homotopies. The purpose of this paper is to prove the following dual result.

1.2 Theorem: Let X be an $(n-1)$-connected based CW-complex with $\dim X \leq k(n-2)+3$, $n \geq 2$. Then X is of the homotopy type of a suspension if and only if X is a co-A_{k-1}-space. Moreover, the homotopy equivalence $f : \Sigma Y \to X$ is a co-A_{k-1}-homomorphism.

1.3 Corollary: A 2-connected space X is of the weak homotopy type of a suspension if and only if it is a co-A_∞-space.

We then use our approach to prove a dual version of Segal's delooping result [11]

1.4 Theorem: Let $X : \triangle \to \mathcal{T}op^*$ be a co-simplicial space such that X_0 is contractible, X_1 is 2-connected and $(\pi_1, \ldots, \pi_n) : \vee_{k=1}^n X_1 \to X_n$ is a homotopy equivalence where $\pi_k : [1] \to [n]$ maps 0 to $i-1$ and 1 to i. Then there is a weak homotopy equivalence $\Sigma RX \to X_1$, where RX is a functorial, thickened version of the topological realization of X.

A result of this type has been formulated by Hopkins [7] with the weaker condition that X_1 is only 1-connected, but remarks by Bousfield [3, (4.9)] indicate that additional assumptions are needed. Instead of the maps π_k Hopkins uses the equivalence $(\iota_1, \ldots, \iota_n) : \vee_{k=1}^n X_1 \xrightarrow{\simeq} X_n$ where $\iota_k : [1] \to [n]$ maps 0 to 0 and 1 to k. A cosimplicial space in his sense induces one in our sense by a base change.

The approach to desupension which we develop here is motivated by potential applications. A fibrant variant of the main result of this paper (see [8], [9]) will be used by the first-named author in a subsequent paper to prove a Haefliger-type embedding theorem in the category of Poincaré duality spaces. Such embedding and desuspension results also play a fundamental role in the original Poincaré surgery program as formulated by Browder et. al. It is hoped that the results of this paper can be adapted so as to complete the Browder program.

Our proofs will make extensive use of homotopy limits and co-limits of cube diagrams and their analysis by Goodwillie [6]. They are in spirit quite different from the proofs of Bernstein-Hilton and Ganea. To stay away from point