VARIETIES OF DISCRETE REPRESENTATIONS OF HYPERBOLIC 3-LATTICES

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ABSTRACT. We survey global properties of the varieties of conjugacy classes of discrete faithful representations of the fundamental groups of closed (or finite volume) hyperbolic 3-manifolds M in linear groups $O(1,m), m \geq 3$, especially for $m=4$ and $m=5$ and their relation to Teichmüller spaces of geometric structures. Of special interest are questions of connectedness of these spaces, cohomology groups $H^1(\pi_1(M), o(1,m))$ (infinitesimal deformations), smooth paths in the varieties and a link with Platonov's conjecture on arithmeticity and super-rigidity. For the last problem, we find a Coxeter group which is non-arithmetic but "hyperbolically super-rigid". Also, in contrast to Teichmüller spaces of hyperbolic surfaces, we show non-connectedness of varieties $\mathcal{T}(\pi_1(M), m), m=4,5$. For $m=4$, this effect is related to an equivariant and everywhere wild embedding of a closed 3-ball into 3-sphere S^3. For $m=5$, such phenomenon has been discovered by a recent author's construction of "twisted" 3-lattice representations in $O(1,5)$. It uses so called block-building method and a topological background related to 2-knots in the 4-sphere. This has also a relation to first constructions of non-trivial circle and 2-plane bundles (with geometric structures) over a closed hyperbolic 3-manifold and wild (even locally wild) quasi-symmetric 2-knots in the 4-sphere.

1. Varieties of group representations

Let $\Gamma = \langle g_1, \ldots, g_d | s_\alpha, \alpha \in A \rangle$ be a finitely generated torsion-free group, $\mathbf{H}$ be a classical simple algebraic group defined over $\mathbb{R}$ with real points H and $\rho_l : \Gamma \to H$ be a fixed embedding of Γ as a lattice in H. If H is not locally isomorphic to $PSL_2(\mathbb{R})$ then Γ is rigid by the Mostow-Margulis rigidity theorem [**M, MA**]. In addition, we suppose that $\mathbf{H}$ is represented in another algebraic group $\mathbf{G}$, $\rho_a : \mathbf{H} \to \mathbf{G}$, also defined over $\mathbb{R}$ where the image of $\mathbf{H}$ is not contained in a proper parabolic subgroup of $\mathbf{G}$. Then the composition of embeddings $\rho_0 = \rho_a \rho_l$ embeds Γ into G as a discrete subgroup (usually no longer a lattice). The main problem is to study the deformation space $\mathcal{R}_G(\Gamma) = \mathrm{Hom}(\Gamma, G)$ (with the algebraic convergence topology) which has the structure of an affine algebraic variety over $\mathbb{R}$ (obtained by embedding it in a product of d copies of G by sending the representation ρ to the d–tuple $(\rho(g_i))_{i=1,\ldots,d}$). Its geometric properties reflect the relationships among the same

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degree representations of Γ. Of most interest is the quotient $\mathcal{S}_G(\Gamma) = R_G(\Gamma)/G$ of this variety by the conjugation action of G. The points of $\mathcal{S}_G(\Gamma)$ correspond to isomorphism classes of simple representations of Γ. A curve in $\mathcal{S}_G(\Gamma)$ through the (isomorphism class $[\rho]$ of) simple representation ρ can be thought of as a deformation of ρ. For a general introduction to representation varieties, see [**LM**].

As applications of representations varieties (see [**GM**]), it is possible to consider a space of locally homogeneous structures on the original locally symmetric space $M = \Gamma\backslash H/K$ where K is a maximal compact subgroup of H. Of most interest are the groups either $SO(1,n)$ or $SU(1,n)$ or $PGL_n(\mathbb{R})$ because the following observation ([**JM**]):

Theorem 1.1. *The representation ρ is locally rigid in G, i.e. its orbit is open in* $\mathrm{Hom}(\Gamma, G)$ *unless H is locally isomorphic to $SO(1,n)$ or $SU(1,n)$.*

It makes sense to consider Γ as a lattice in n-dimensional hyperbolic space (either real or complex). Due to the fact that the case of complex hyperbolic structures is much more rigid than the real one ([**GMI**]), our basic object will be the fundamental group $\Gamma = \pi_1(M)$ of a closed (or finite volume) hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$, or a hyperbolic n-lattice $\Gamma \subset \mathrm{O}^\circ(1,n) = H, \quad n \geq 3$.

2. Representations of hyperbolic lattices and geometric structures

Now we assume that Γ is a hyperbolic lattice (in a semi-simple Lie group $H = \mathrm{O}^\circ(1,n)$ of real rank 1, $n \geq 3$) and we have its representations in another semi-simple Lie group G (especially, $G = \mathrm{O}^\circ(1,m)$, $m \geq 4$).

Inside of $\mathcal{R}_m(\Gamma) = \mathcal{R}_{\mathrm{O}^\circ(1,m)}(\Gamma)$ we have two sequential subspaces of faithful representations with discrete (correspondingly, discontinuous somewhere at $\partial\mathbb{H}^m = S^{m-1}$) images. $\mathrm{O}^\circ(1,m)$ acts on $\mathcal{R}_m(\Gamma)$ by conjugation leaving both subspaces invariant. We denote their quotient spaces as

$$\mathcal{H}_m(\Gamma), \quad \mathcal{T}_{m-1}(\Gamma) \subset \mathcal{R}_m(\Gamma)/\,\mathrm{O}^\circ(1,m), \tag{2.1}$$

correspondingly.

There is a natural identification of the biggest space $\mathcal{H}_m(\Gamma)$ with the space of n-dimensional hyperbolic structures on Γ. An element of this space is determined by an equivalence class of pairs $\{N, \phi\}$ where N is an m-dimensional hyperbolic manifold and $\phi : \Gamma \to \pi_1(N)$ is an isomorphism (up to isometries of manifolds N and inner automorphisms of Γ).

As an example, if $\Gamma = \pi_1(S_g)$ for a closed surface of genus $g > 1$, then $\mathcal{H}_2(\Gamma)$ is the Teichmüller space of Γ and homeomorphic to $\mathbb{R}^{6g-6}$. Another example of $\Gamma = \pi_1(M)$ for a hyperbolic closed n-manifold M provides the space $\mathcal{H}_n(\Gamma)$ which degenerates to a point due to the Mostow-Margulis rigidity ([**M**], [**MA**]):

Theorem 2.1. *Let M_1, M_2 be closed hyperbolic n-manifolds with isomorphic fundamental groups. Then M_1 and M_2 are isometric.*

Similarly, via holonomy, we have a natural identification of the space $\mathcal{T}_{m-1}(\Gamma)$ with the (Teichmüller) space of uniformizable $(m-1)$-dimensional conformal structures on Γ (cf. [**A1**], [**A3**]). An element of this space is determined by a pair