MAXIMAL PARABOLIC SUBGROUPS IN CLASSICAL GROUPS ARE OF MODALITY ZERO

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Abstract. The principal aim of this paper is to show that every maximal parabolic subgroup P of a classical reductive algebraic group G operates with a finite number of orbits on its unipotent radical. This is a consequence of the fact that each parabolic subgroup of a group of type A_n whose unipotent radical is of nilpotent class at most two has this finiteness property.

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field K of characteristic 0. Let $P = LP_u$ be a parabolic subgroup of G with P_u the unipotent radical of P and L a Levi complement in P. As P_u is a normal subgroup of P, the latter group acts on the former via conjugation and on the Lie algebra of P_u via the adjoint action.

In [6] the general problem was posed to determine each parabolic subgroup P of a reductive algebraic group G that has a finite number of orbits on P_u. The cases for Borel and semisimple rank one parabolic subgroups were classified in [5] and [6] respectively.

In [6, Problem 5.6] the question was raised whether every maximal parabolic subgroup of G acts with a finite number of orbits on its unipotent radical. For classical groups the answer is affirmative:

Theorem 1.1. *Let G be a classical simple algebraic group and P a maximal parabolic subgroup of G. Then P acts on P_u with a finite number of orbits.*

Remark 1.2. The analogous statement of Theorem 1.1 for exceptional groups fails. There are two conjugacy classes of maximal parabolic subgroups P in E_8 such that P acts with an infinite number of orbits on P_u; see [9, Remark 3.3].

Theorem 1.1 is a consequence of

Theorem 1.3. *Let G be of type A_n and P a parabolic subgroup of G. If the length of the descending central series of P_u is at most two, then P acts on P_u with a finite number of orbits.*

Finally, we prove a generalization of Theorems 1.1 and 1.3:

Theorem 1.4. *Let G be a classical simple algebraic group and P a parabolic subgroup of G. If the length of the descending central series of P_u is at most two, then P acts on P_u with a finite number of orbits.*

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As in [6] and [9] we take the more general point of view in terms of the *modality* of an algebraic group acting morphically on an algebraic variety. Recall from [12] that for an algebraic group R acting morphically on an algebraic variety X the *modality* of the action, $\operatorname{mod}(R : X)$, is defined as

$$\operatorname{mod}(R : X) = \max_{Z} \min_{z \in Z} \operatorname{codim}_{Z} R^0 \cdot z,$$

where Z runs through all irreducible R^0-invariant subvarieties of X, and R^0 denotes the identity component of R. Informally, $\operatorname{mod}(R : X)$ is the maximal number of parameters on which a family of R-orbits on X may depend.

For a parabolic subgroup P of an algebraic group we call $\operatorname{mod} P := \operatorname{mod}(P : P_u)$ the *modality* of P, cf. [6]. Observe that the modality of P is 0 precisely when P admits only a finite number of orbits on P_u.

In Section 2 we first recall some basic results from [6] and [9] and apply them to our situation in Section 3 in showing that Theorem 1.1 follows from Theorem 1.3. In the next section we outline a proof of Theorem 1.3 followed by a section where we prove Theorem 1.4.

For an algebraic group R we often denote its Lie algebra by $\operatorname{Lie} R$ or simply by the corresponding lower case gothic letter $\mathfrak{r}$ and its unipotent radical by R_u. The Lie algebra of R_u is denoted by $\mathfrak{r}_u$. Throughout the identity element of an algebraic group is denoted by e.

The descending central series $\{C^i R \mid i \geq 0\}$ of R is defined as usual: $C^0 R := R$ and $C^{i+1} R := [C^i R, R]$ for $i \geq 0$. If $C^i R = \{e\}$ for some $i \geq 0$, then R is nilpotent. In that case the *length* of the descending central series of R, denoted by $\ell(R)$, is the smallest integer m such that $C^m R = \{e\}$, i.e., $\ell(R) = m$; equivalently, R is nilpotent of class m.

If R acts morphically on an algebraic variety X, we denote by $R \cdot x$ the orbit of x in X under the action of R and if Z is a subset of X, then $R \cdot Z$ is the union of all orbits $R \cdot z$ in X, where $z \in Z$.

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. As usual, for a root $\alpha \in \Psi$ we denote the corresponding root space in $\mathfrak{g}$ by $\mathfrak{g}_\alpha$. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called a standard parabolic subgroup. For a subset J of Π we denote by P_J the standard parabolic subgroup corresponding to J such that $P_\varnothing = B$.

As every parabolic subgroup P of G is conjugate to a standard one, and since $\ell(P_u)$ and $\operatorname{mod} P$ are invariant under conjugation by elements from G, we may if we wish assume that the parabolic subgroups under consideration are standard.

Because a reductive algebraic group is the direct product of simple ones over its center, and similarly for parabolic subgroups, we can reduce to the case of a simple algebraic group in our study of parabolic subgroups of modality zero. Henceforth we assume that G is a connected, simply connected simple algebraic group, unless stated otherwise.

Since the characteristic of K is 0, the exponential mapping is a P-equivariant isomorphism between $\operatorname{Lie} P_u$ and P_u. This allows us to work on $\operatorname{Lie} P_u$ instead of P_u, whenever convenient.