SETS OF INTEGERS WITH PAIRWISE COMMON DIVISOR AND A FACTOR FROM A SPECIFIED SET OF PRIMES

by

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1. Introduction and main results

Whenever possible we keep the notation of [3]. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}^*$ is the set of positive squarefree numbers. $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of primes and p_s denotes the s–th prime.

For two numbers $u, v \in \mathbb{N}$ we write $u|v$ (resp. $u \nmid v$) iff u divides v (resp. u does not divide v). $[u, v]$ stands for the least common multiple of u and v . (u, v) is the largest common divisor of u, v and we say that u and v have a common divisor if $(u, v) > 1$. $\langle u, v \rangle$ denotes the interval $\{x \in \mathbb{N} : u \le x \le v\}$ and $(u, v\rangle$ denotes the left–open interval $\{x \in \mathbb{N} : u < x \le v\}$.

For any set $A \subset \mathbb{N}$ we introduce $A(n) = A \cap \langle 1, n \rangle$ and $|A|$ as cardinality of A . The set of multiples of A is

$$M(A) = \{m \in \mathbb{N} : a|m \text{ for some } a \in A\}.$$

For $u \in \mathbb{N}, u \neq 1, p^+(u)$ (resp. $p^-(u)$) denotes the largest (resp. the smallest) prime factor of u .

For any $y \in \mathbb{N}$, $\pi(y) = |\mathbb{P}(y)|$ denotes the counting function of primes. For any subset of primes $T \subset \mathbb{P}$, and $u \in \mathbb{R}^+$ we set

$$\phi(u, T) = \big\{x \in \mathbb{N}(u) : (x, p) = 1 \text{ for all } p \in T\big\}.$$

We note that always $\{1\} \in \phi(u, T)$ for all $T \subset \mathbb{P}$, $u \ge 1$.

Finally, for a set $A = \{a_1, \dots, a_m\}$ of ordered numbers $a_1 < a_2 < \dots < a_m$ we also just write $A = \{a_1 < a_2 < \dots < a_m\}$.

P. Erdös and R. Graham (see [1], [2]) posed the following problem:

Let $1 < a_1 < a_2 < \dots < a_k = n$, $(a_i, a_j) \neq 1$. What is the maximal value of k ? We denote it by $g(n)$.

While in [1] the problem was stated unfortunately with many confusing misprints, in [2] one can find the following conjecture: $g(n)$ equals either $\frac{n}{p^-(n)}$ or the number of integers of the form $2 \cdot t$, $t \le \frac{1}{2}n$, $(t, n) \neq 1$.

However, it is easy to find a counterexample for this assertion and we informed Erdös about this during his visit in Bielefeld in the year 1992. He then came up with the following formulation:

Conjecture 1. Let $n = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \dots q_r^{\alpha_r}$, $\alpha_i \ge 1$, $q_i \in \mathbb{P}$, and $q_1 < q_2 < \dots < q_r$, then

$$g(n) = \max_{1 \le j \le r} |M(2q_1, 2q_2, \dots, 2q_j, q_1 \dots q_j) \cap \mathbb{N}(n)|.$$

We consider a more general and seemingly more natural problem:

Let $Q = \{q_1 < \dots < q_r\} \subset \mathbb{P}$ be any finite set of primes and let $A = \{a_1 < a_2 < \dots < a_k\} \subset \mathbb{N}(n)$, be a set such that for all $1 \le i, j \le k$

$$(a_i, a_j) \neq 1 \tag{1.1}$$