SOME UNIVERSAL RESULTS ON THE
BEHAVIOR OF INCREMENTS OF PARTIAL SUMS

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Abstract. We establish very general one-sided results on the lim sup behavior of increments of suitably normalized partial sums of i.i.d. random variables. Our main results apply to arbitrary nondegenerate positive random variables which need not have any finite moments. As a corollary we can show that such results also hold for not necessarily positive random variables whose negative parts have finite moment-generating functions.

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1. Introduction. Let $X, X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) nondegenerate random variables with distribution function F. As usual, set $S_0 = 0, S_n = X_1 + \ldots + X_n, n \geq 1$, and let $Lt := \log(max(t, e)), LLt := L(Lt), t \geq 0$. Then it follows from the classical Hartman-Wintner LIL that one has under the assumptions $EX = 0, EX^2 = 1$, with probability one,

$$\limsup_{n \to \infty} |S_n|/(2nLLn)^{1/2} = 1. \tag{1.1}$$

It is also known that the above moment assumptions are necessary for (1.1) to hold. Much more general LIL results, however, are attainable, if one considers one-sided versions of (1.1) and uses different centering and norming sequences (see, for instance, Klass (1976), Mason (1994), and Pruitt (1981)).

Mason (1994) has recently shown that if X is an arbitrary nondegenerate positive random variable, then one can find centering constants $\{\mu_n\}$ and norming constants $a_n \nearrow \infty$ such that with probability one,

$$0 \leq \limsup_{n \to \infty} \{n\mu_n - S_n\}/a_n \leq 2^{1/2}. \tag{1.2}$$

Moreover, it follows from a result of Einmahl and Mason (1994) that the lim sup in (1.2) is positive, whenever X is in the Feller class. Recall that a random variable is in the Feller class, if one can find centering constants $\{\delta_n\}$ and norming constants $\{c_n\}$ such that

(1.3) $(S_n - \delta_n)/c_n$ is tight with nondegenerate subsequential limits.

Note in particular that any random variable in the domain of attraction to a stable law of index $\alpha \in (0, 2]$ belongs to the Feller class.

The purpose of the present paper is to investigate whether one can obtain similar general results on the behavior of increments of partial sums of i.i.d. random variables. Our starting point is a result of Csörgő and Révész (1979), the exact formulation of which requires some additional notation. Let $0 < \kappa_n \leq n$ be a non-decreasing sequence of real numbers satisfying

(1.4) κ_n/n is non-increasing, and

(1.5) $\kappa_n/\log n \to \infty$ as $n \to \infty$.

Set $k_n := [\kappa_n]$, $n \geq 1$, where $[x]$ denotes the integer part of $-\infty < x < \infty$, and let for $1 \leq k \leq n$,

$$M_n(k) := \max_{0 \leq i \leq n-k} \max_{0 \leq j \leq k} |S_{i+j} - S_i|.$$