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Green's Theorem on Hall Algebras

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ABSTRACT. Let k be a finite field and Λ a hereditary finitary k-algebra. Let $\mathcal{P}$ be the set of isomorphism classes of finite Λ-modules. We define a multiplication on the $\mathbb{Q}$-space with basis $\mathcal{P}$ by counting the number of submodules U of a given module V with prescribed isomorphism classes both of V/U and U. In this way we obtain the so called Hall algebra $\mathcal{H}{=}\mathcal{H}(\Lambda,\mathbb{Q})$ with coefficients in $\mathbb{Q}$. Besides $\mathcal{H}$, we are also interested in the subalgebra $\mathcal{C}$ generated by the the subset I of all isomorphism classes of simple Λ-modules; this subalgebra is called the corresponding composition algebra, since it encodes the number of composition series of all Λ-modules.

Recently, J. A. Green has introduced on $\mathcal{H}$ (and on $\mathcal{C}$) a comultiplication δ so that it becomes nearly a bialgebra. Here, "nearly" means that $\delta\colon \mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ is in general not an algebra homomorphism for the usual (componentwise) multiplication on $\mathcal{H}\otimes\mathcal{H}$; instead, there is a slightly twisted multiplication on $\mathcal{H}\otimes\mathcal{H}$ which has to be considered. The appearance of this twist should not be considered as a disadvantage: on the contrary, it is the main ingredient for Drinfeld and Lusztig in order to define the positive part U^+ of a quantum group for an arbitrary Cartan datum. It follows that U^+ is isomorphic to a twisted generic composition algebra.

Green's compatibility theorem for multiplication and comultiplication on $\mathcal{H}$ is expressed in a marvelous formula which deals with pairs of submodules of a given module. This formula should be considered as a detailed investigation of the $3{\times}3$ Lemma in homological algebra.

This is the written version of two lectures given at the workshop preceding ICRA VII, held at U.N.A.M., Mexico, D.F. We present the complete proof of Green's formula and outline its consequences, following closely the presentation in the original paper [G1].

Part I. Green's Formula

Let Λ be a ring. Let $\mathcal{P}$ be the set of isomorphism classes of Λ-modules of finite cardinality. For every $\alpha\in\mathcal{P}$ let V_α be a representative in α. We denote by 0 both the zero module and its isomorphism class.

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We recall that Λ is said to be *finitary,* provided $\operatorname{Ext}^1(M,N)$ is a finite set for any pair M,N of finite Λ-modules (clearly, it is sufficient to check this for finite, simple Λ-modules M,N). Throughout these notes, we usually will assume that the ring Λ is finitary.

Also, Λ is said to be hereditary, provided $\operatorname{Ext}^2(M,N)=0$ for all Λ-modules M,N. It is well-known that an algebra is hereditary if and only if submodules of projective modules are projective – and this explains the name.

1. The formula

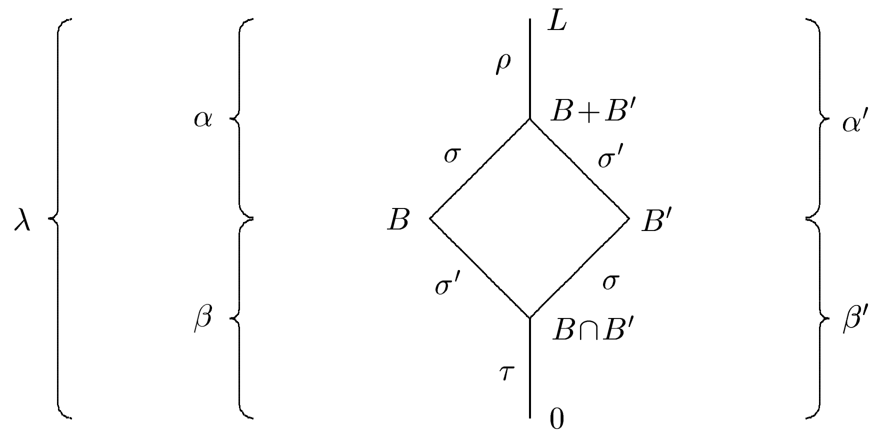
Given $\alpha\in\mathcal{P}$, let a_α be the order of the automorphism group of V_α.

Given α,β,λ in $\mathcal{P}$, let $g^\lambda_{\alpha\beta}$ be the number of submodules B of V_λ such that V_λ/B is isomorphic to V_α, whereas B is isomorphic to V_β.

Theorem (J. A. Green). *Let k be a finite field and Λ a k-algebra which is hereditary and finitary. Let $\alpha,\beta,\alpha',\beta'$ be elements of $\mathcal{P}$. Then*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'}\, a_\lambda^{-1} = \sum_{\rho,\sigma,\sigma',\tau} \frac{|\operatorname{Ext}^1(V_\rho,V_\tau)|}{|\operatorname{Hom}(V_\rho,V_\tau)|}\, g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau .$$

Green's theorem exhibits a universal formula for pairs of submodules: We consider submodules B and B' of a module L, where $L/B\in\alpha$, $L/B'\in\alpha'$, $B\in\beta$ and $B'\in\beta'$ (these four isomorphism classes $\alpha,\beta,\alpha',\beta'$ are given). The left hand side counts the numbers of such pairs of submodules in arbitrary modules $L=V_\lambda$, always weighted with a_λ^{-1}. The right hand side involves the four subfactors $L/(B+B')$, $B/(B\cap B')$, $B'/(B\cap B')$, and $B\cap B'$; their isomorphism classes are denoted by ρ,σ',σ and τ, respectively.



We stress that the right hand side of the formula only involves small modules (namely modules which occur as submodules or factor modules of the given modules $V_\alpha,V_\beta,V_{\alpha'},V_{\beta'}$), whereas the left hand side deals with modules V_λ, where $\dim_k V_\lambda = \dim_k V_\alpha + \dim_k V_\beta$.