Compactness properties of locally compact groups

Herbert Abels and Andreas Tiemeyer

For discrete groups there are finiteness properties FP_n and F_n: A group Γ is said to be of type FP_n if there is a projective resolution over $\boldsymbol{Z}\Gamma$ of the trivial $\boldsymbol{Z}\Gamma$–module $\boldsymbol{Z}$ with finitely generated modules in dimension $\leq n$. If there is an Eilenberg–MacLane space $K(\Gamma, 1)$ with finite n–skeleton, then Γ is said to be of type F_n. The properties F_1 and FP_1 are both equivalent with finite generation; F_2 is equivalent with finite presentability and implies FP_2. Whether F_2 and FP_2 are in fact equivalent is still an open question. A group is of type F_n if and only if it is of type FP_n and (for $n \geq 2$) of type F_2 (see e.g. [Se], [Wa 1], [Wa 2]).

In this paper we introduce compactness properties C_n and CP_n for locally compact groups, which generalize these finiteness properties.

In section 1 we give the basic definitions. Starting point is a theorem of K. Brown ([Br 2, thm 2.2]) giving a necessary and sufficient condition for a group to be of type FP_n.
Section 2 is devoted to the investigation of the property C_n for $n \leq 2$. We prove that there is (as in the discrete case) a group–theoretical interpretation: The property C_1 is equivalent with compact generation, C_2 is equivalent with compact presentability.
In section 3 we prove some basic results about compactness properties. In some sense this section shows that the compactness properties are reasonably defined. Specifically it is proved that the compactness properties remain unchanged by passing to a cocompact subgroup or by taking the quotient by a compact normal subgroup and that they are preserved under direct products.

In a different publication the second named author will apply compactness properties to S–arithmetic groups and prove a local–global principle for the finiteness properties of these groups.

1 ind–objects

In this chapter we give the definition of the compactness properties C_n and CP_n. To do this we use the notion of an ind–object in a given category (following Grothendieck) as well as some familiar constructions in the category of simplicial sets, which we recall. It will be an immediate consequence of [Br 2, thm. 2.2] that for discrete groups finiteness properties and compactness properties coincide.

1.1 A Hurewicz theorem for ind–spaces

Let $\mathcal{C}$ be any category. Then the category of ind–objects in $\mathcal{C}$, denoted by $\underline{\text{ind}}-\mathcal{C}$, is given by the following definitions (see also [Gr, 8.2]):

An object of $\underline{\text{ind}}-\mathcal{C}$ is a functor $\underline{A} : I \to \mathcal{C}$, where I is a filtered set regarded as a category. We often write A_i instead of $\underline{A}(i)$ and $\{A_i\}_{i \in I}$ instead of $\underline{A}$. In the terminology of [Gr] all these ind–objects are "essentially small".

A morphism in this category, however, is *not* a natural transformation of functors or some version of it, which takes into account the fact that the Category I is variable, but

$$\mathrm{Hom}_{\underline{\text{ind}}-\mathcal{C}}(\{A_i\}_{i \in I}, \{B_j\}_{j \in J}) = \lim_i \ \mathrm{colim}_j \ \mathrm{Hom}_{\mathcal{C}}(A_i, B_j).$$

In more detail, a morphism in $\underline{\text{ind}}-\mathcal{C}$, say from $\underline{A} : I \to \mathcal{C}$ to $\underline{B} : J \to \mathcal{C}$, is given by
(i) An (arbitrary) map $\alpha : I \to J$
(ii) A collection of $\mathcal{C}$–morphisms $\{\phi_i\}_{i \in I}$, where $\phi_i : A_i \to B_{\alpha(i)}$
For these data to give rise to a morphism in $\underline{\text{ind}}-\mathcal{C}$ we require: For all i, $i' \in I$ with $i < i'$ there is a $j \in J$ with $\alpha(i) < j > \alpha(i')$, such that the diagram

$$\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[d] & B_{\alpha(i)} \arrow[dr] & \\
A_{i'} \arrow[r, "\phi_{i'}"] & B_{\alpha(i')} \arrow[r] & B_j
\end{tikzcd}$$

commutes, where the unmarked arrows are the structure maps of I and J.

Two such data $\Phi = (\alpha, \{\phi_i\}_{i \in I})$ and $\Psi = (\beta, \{\psi_i\}_{i \in I})$ give rise to the same morphism, if ϕ_i and ψ_i finally coincide. More precisely we have $\Phi = \Psi$, if and only if for each $i \in I$ there is a $j \in J$ with $\alpha(i) < j > \beta(i)$ such that the following diagram commutes:

$$\begin{tikzcd}
& B_{\alpha(i)} \arrow[dr] & \\
A_i \arrow[ur, "\phi_i"] \arrow[dr, "\psi_i"'] & & B_j \\
& B_{\beta(i)} \arrow[ur] &
\end{tikzcd}$$