A local–global principle for finiteness properties of S–arithmetic groups over number fields

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1 Introduction

In [AT] compactness properties C_n for locally compact groups were introduced. In this paper the following theorem is proved:: Let Γ be an S–arithmetic subgroup of a linear algebraic group G over a number field. Then Γ is of type F_n if and only if for every non–Archimedian $p \in S$ the p–adic completion G_p is of type C_n. This generalizes a result of Kneser ([Kn]), who proved it for $n = 1,\ 2$. An immediate consequence is that one only has to deal with one prime at a time in the investigation of finiteness properties of S–arihmetic groups. Also, one can use the strong results on algebraic groups over local fields.

Furthermore it is shown that reductive groups over locally compact fields of characteristic 0 are of type C_∞. So the local–global principle gives a new proof of a result of Borel and Serre ([BS]), stating that S–arithmetic subgroups of reductive groups have all finiteness properties. The new proof, however, is much easier.

We begin with summarizing the basic definitions of [AT]:
Let G be a locally compact group and suppose X is a G–space. The free simplicial set on X, denoted by EX, is the simplicial set whose set of n–simplices is

$$(EX)_n = X^{n+1}.$$

The face maps are given by cancelling, the degeneracy maps by doubling one factor. Then EX is contractible and G acts on the geometric realization $|EX|$. For $C \subseteq X$ denote by $G \cdot EC$ the subcomplex of EX defined as follows: $\sigma = (x_0, \ldots, x_n)$ is a simplex of $G \cdot EC$ if there is a $g \in G$ such that $\{x_0, \ldots, x_n\} \subseteq gC$.

Definition 1.1 *We say that the G–space X satisfies the condition P_n (or, more briefly, that $P_n(X, G)$ holds) if for every compact $C \subseteq X$ there is a compact $D \subseteq X$ with $C \subseteq D$, such that the inclusion*

$$i : G \cdot EC \to G \cdot ED$$

induces the zero map on π_i for all $i < n$.

The group G is said to be of type C_n, if the G–space G satisfies the condition P_n.

The property C_n is a generalization of the finiteness property F_n. By using homology groups instead of homotopy groups one obtains the property CP_n, which generalizes the property FP_n. In this paper we will only be concerned with the properties C_n. However, all the results are also valid for CP_n by almost literally the same proofs.

2 Borel's finiteness theorem

To begin with we establish some notation which will remain fixed throughout the rest of the paper. It is mostly adopted form [Bo, chapter 1]. Denote by

k	an algebraic number field of finite degree
o	an ring of integers of k
V	an set of primes of k
P	an set of non–Archimedian primes
k_v	an completion of k with respect to the valuation v
o_p	an ring of p–adic integers of k_p for $p \in P$

For $v \in V - P$ let $o_v = k_v$ and for $S \subseteq V$ let $o(S)$ be the ring of S–integers of k, i.e. $o(S) = \{x \in k : x \in o_v \text{ for } v \in V - S\}$. If $S \subseteq V - P$, then $o(S) := o$.

Let G be a linear algebraic group, which we will assume to be embedded in a general linear group. There is no loss of generality in this assumption, because the local–global principle we are going to proove (theorem 3.1) does not depend on the chosen embedding: The groups on its local side are k_v–points of G, these do not at all depend on the embedding. On the global side of the principle we have the S–arithmetic group $G_{o(S)}$, which depends on the embedding, but only up to commensurability, so the embedding has no effect on its finiteness properties.

For a subring of an overfield of k let G_B be the group of elements of G with coefficients in B and with determinant a unit of B. We write G_v instead of G_{k_v} and G_A for the adele group of G. The group G_v is locally compact and for $p \in P$ the group G_{o_p} is open and compact in G_p. For $S \subseteq V$ let

$$G_S := \prod_{v \in S} G_v \ , \quad G_{A(S)} := \prod_{v \in S} G_v \times \prod_{v \in V-S} G_{o_v} \ .$$

If $S \subseteq V$ is finite, then G_S and $G_{A(S)}$ are locally compact. Let $\pi_S : G_A \to G_S$ be the canonical projection and let $j_S : G_S \to G_A$ be the injection given by the condition that the components of the image of j_S outside S are equal to the neutral element.

In case $S = V - P$ we write G_o, G_∞, G_A^∞, π_∞ and j_∞ instead of $G_{o(S)}$, G_S, $G_{A(S)}$, π_S and j_S. With every $a \in G_k$ there is associated an element of G_A,