Frobenius functions on translation quivers

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Abstract. Frobenius functions are integral valued functions given on vertices of translation quivers and satisfying certain subadditivity conditions. Typical examples are the length function and the dimension function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra. In our paper we study in detail Frobenius functions on the translation quivers $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty^\infty$ and some related ones. In particular we show that there is a one-to-one correspondence between Frobenius functions on the stable tube $T(n)$ and Frobenius functions on the wing $W(n)$, and we classify them using certain related combinatorial structures.

We denote by $W(n)$ the wing of order n (it is the Auslander–Reiten quiver of the ring $A_n(k)$ of upper triangular $n \times n$-matrices over a field k) and by $T(n) = \mathbb{Z}A_\infty/\langle \tau^n \rangle$ the stable tube of rank n (it is the Auslander–Reiten quiver of the locally nilpotent representations of the cyclic quiver with n vertices). Note that $T(n)$ has n full subquivers isomorphic to $W(n)$ which contain all vertices of the mouth; we call them the *maximal wings of* $T(n)$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver (without multiple arrows) and let $f : \Gamma_0 \longrightarrow \mathbb{Z}$ be a map. For any non-projective vertex z of Γ, define $\delta(z) = \delta_f(z) = f(z) + f(\tau z) - \sum_{y \to z} f(y)$. A *Frobenius function* f on Γ is a map $f : \Gamma_0 \longrightarrow \mathbb{Z}$ such that for any non-projective vertex z of Γ with $\delta(z) \neq 0$, we have both $f(z) < \delta(z)$ and $f(\tau z) < \delta(z)$. Note that any non-negative Frobenius function is subadditive, that is, $\delta(z) \geq 0$ for any non-projective vertex. In dealing with a Frobenius function f on Γ, and a non-projective vertex z with $\delta(z) \neq 0$, we say that the mesh ending in z is an *incomplete mesh*. Typical examples of positive Frobenius functions are given by the dimension function or the length function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra; the incomplete meshes being those, where an indecomposable projective module has been removed.

Our aim is a detailed study of positive Frobenius functions on wings, stable tubes and the translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$. By using certain combinatorial

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structures we give a classification of these functions. In the forthcoming second part of the paper algebraic representations will be given for some of these results. The results were reported at the Sixth International Conference on Representations of Algebras, held at Ottawa in 1992.

We would like to mention that many of the ideas used here can be traced back to [GR]. For unexplained notation, we refer to [R].

1. The main results

There is a strong interrelation between Frobenius functions on $W(n)$ and $T(n)$.

THEOREM 1.1. *If g is a positive Frobenius function on $T(n)$, then there exists a maximal wing of $T(n)$ which contains all incomplete meshes. Conversely, if U is a maximal wing of $T(n)$ then any function $f : U_0 \longrightarrow \mathbb{N}_1 = \{1, 2, 3, \ldots\}$ has a unique extension to a function $\bar{f} : T(n)_0 \longrightarrow \mathbb{N}_1$ which is additive on all meshes which are not contained in U. Such an extension $\bar{f}$ is a positive Frobenius function if and only if f is.*

Special cases of this statement have been considered by Erdmann (see [E]) in her study of tame symmetric algebras.

A consequence of the above theorem is that any positive Frobenius function on $T(n)$ is uniquely determined by its restriction to an appropriate maximal wing $W(n)$. Thus we may focus our attention to positive Frobenius functions on wings.

Let us denote the projective vertices of $W(n)$ by $\mathbf{p}_1$, $\mathbf{p}_2$, …, $\mathbf{p}_n$, indexed so that there are arrows $\mathbf{p}_1 \to \mathbf{p}_2 \to \cdots \to \mathbf{p}_n$. Given a function $f : W(n) \longrightarrow \mathbb{Z}$, take $f_i = f(\mathbf{p}_i)$ and consider the vector $(f_1, \ldots, f_n)$. We call a vector $(f_1, \ldots, f_n)$ with entries in $\mathbb{N}_1$ *binary* if whenever $f_i = f_j$ for some $i < j$, then there exists an index ℓ such that $i < \ell < j$ and $f_\ell < f_i$. A binary vector $f = (f_1, \ldots, f_n)$ yields an embedded rooted binary tree $B(f)$ with n vertices as follows. The empty vector (for $n = 0$) will correspond to the empty tree. Given a binary vector $f = (f_1, \ldots, f_n)$, let f_t be the minimal coordinate (by definition of a binary vector, t is uniquely defined). Let $g = (f_1 - f_t, \ldots, f_{t-1} - f_t)$, and $h = (f_{t+1} - f_t, \ldots, f_n - f_t)$. We take the vertex with index t as root, and, in case g or h are non-empty, we attach to t the trees $B(g)$ and $B(h)$ so that the root of $B(g)$ is the upper left neighbor and the root of $B(h)$ is the upper right neighbor of t. Note that any embedded rooted binary tree can be obtained in this way. Binary vectors f and f' with $B(f) = B(f')$ will be said to be *equivalent*. For example, the binary vectors $(7, 3, 5, 4, 5, 7)$ and $(2, 1, 3, 2, 3, 4)$ are equivalent, the corresponding binary tree is: