The complete intersection theorem for systems of finite sets

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1 Historial background and the new theorem

We are concerned here with one of the oldest problems in combinatorial extremal theory.

It is readily described after we have made a few conventions. $\mathbb{N}$ denotes the set of positive integers and for $i, j \in \mathbb{N}$, $i < j$, the set $\{i, i+1, \dots, j\}$ is abbreviated as $[i, j]$. For $k, n \in \mathbb{N}$, $k \leq n$, we set

$$2^{[n]} = \{F : F \subset [1, n]\}, \qquad \binom{[n]}{k} = \{F \in 2^{[n]} : |F| = k\}. \tag{1.1}$$

A system of sets $\mathcal{A} \subset \binom{[n]}{k}$ is called t*–intersecting*, if

$$|A_1 \cap A_2| \geq t \text{ for all } A_1, A_2 \in \mathcal{A}, \tag{1.2}$$

and $I(n, k, t)$ denotes the set of all such systems.

The investigation of the function

$$M(n, k, t) = \max_{\mathcal{A} \in I(n,k,t)} |\mathcal{A}|, \qquad 1 \leq t \leq k \leq n, \tag{1.3}$$

and the structure of maximal systems was initiated by Erdös, Ko, and Rado. According to [1] they proved already in the year 1938 the following theorem, which was published only in 1961 in the famous paper [2].

Theorem EKR.

For $1 \leq t \leq k$ and $n \geq n_o(k, t)$ (suitable)

$$M(n, k, t) = \binom{n-t}{k-t}. \tag{1.4}$$

Clearly, the system

$$\mathcal{A}(n, k, t) = \left\{A \in \binom{[n]}{k} : [1, t] \subset A\right\} \tag{1.5}$$

has cardinality $\binom{n-t}{k-t}$ and is therefore optimal for $n \geq n_o(k, t)$.

The smallest $n_o(k, t)$ for which this is the case has been determined by Frankl [3] for $t \geq 15$ and subsequently by Wilson [4] for all t:

$$n_o(k, t) = (k - t + 1)(t + 1). \tag{1.6}$$

Moreover, for $n > (k - t + 1)(t + 1)$ there is — up to obvious permutations on the ground set $[1, n]$ — only one optimal system.In the present paper we settle all the remaining cases

$$n < (k - t + 1)(t + 1).$$