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Cones

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Dedicated to the Memory of Maurice Auslander

ABSTRACT. A cone is the full translation subquiver of $\mathbb{Z}A_\infty$ given by the predecessors of some fixed vertex on the boundary. We consider an abelian category $\mathcal{A}$ with Auslander-Reiten sequences and we denote its Auslander-Reiten quiver by Γ. We are going to present a criterion for an indecomposable object X in $\mathcal{A}$ in order that the predecessors of X in Γ form a cone. As a consequence, we will see that for suitable graded algebras, any stable Auslander-Reiten component which contains a Koszul module is of the form $\mathbb{Z}A_\infty$.

Let k be a field. Let $\mathcal{A}$ be an abelian k-category (with finite dimensional Hom-sets) such that every object has finite length. Assume that $\mathcal{A}$ has projective covers and injective envelopes, thus $\mathcal{A}$ has Auslander-Reiten sequences. We denote by τ the corresponding Auslander-Reiten translation.

We are interested in Auslander-Reiten components of the form $\mathbb{Z}A_\infty$. The full translation subquiver of $\mathbb{Z}A_\infty$ given by all predecessors of a fixed vertex on the boundary is said to be a *cone*.

THEOREM. *Let X be an indecomposable object in $\mathcal{A}$ such that for all $t \in \mathbb{N}_0$, the object $\tau^t X$ is defined and that none of the objects $\tau^t X$ with $t \geq 1$ is simple. Let Γ be the Auslander-Reiten component of $\mathcal{A}$ containing $[X]$. Assume that*

$$\mathrm{Hom}(\tau^t X, \tau^s X) = 0 \quad \textit{for all} \quad 0 \leq s < t.$$

Then the set of predecessors of $[X]$ in Γ is a cone.

We note the following: if we assume, in addition, that Γ is a stable translation quiver, then Γ is of the form $\mathbb{Z}A_\infty$.

Proof. Let $X_{ii} = \tau^i X$ for alle $i \in \mathbb{N}_0$. We are going to construct indecomposable modules X_{st} with $0 \leq s \leq t$ such that the modules X_{ij} with $s \leq i \leq j \leq t$ form a wing in Γ.

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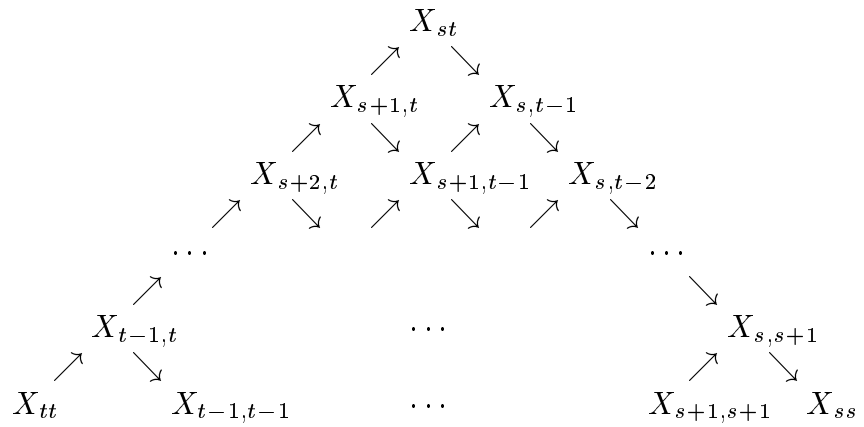

This of course implies that there exists an irreducible monomorphism

$$u_{s+1,t}\colon X_{s+1,t} \to X_{st}$$

with cokernel X_{ss}. Consequently X_{st} has a filtration with factors (going down) of the form $X_{ss}, X_{s+1,s+1}, \dots, X_{tt}$.

Let us assume that we have constructed the modules X_{st} for $0 \le t-s \le n$, as well as irreducible maps $u_{s+1,t}\colon X_{s+1,t} \to X_{st}$ with cokernel X_{ss}, for $1 \le t-s \le n$. First, let us show that τX_{st} is isomorphic to $X_{s+1,t+1}$.

By definition, this is true for $s = t$. Thus, let us assume $s < t$. First of all, we claim that X_{st} cannot be indecomposable projective. For the proof, we consider the length d_{st} of the top of X_{st} and we show that $d_{st} \ge 2$ for $s < t$. Consider the Auslander-Reiten sequence

$$0 \to X_{s+1,t} \to X_{st} \oplus X_{s+1,t-1} \to X_{s,t-1} \to 0$$

(where $X_{s+1,t-1} = 0$ for $t = s+1$). Since $X_{s+1,t}$ is not simple, the top of the middle term is isomorphic to the direct sum of the tops of the outer terms, thus

$$d_{st} = d_{s+1,t} + d_{s,t-1} - d_{s+1,t-1}.$$

Using induction, we see that $d_{st} = \sum_{i=s}^{t} d_{ii}$. But this implies $d_{st} \ge t-s+1$.

There is an irreducible map $X_{s+1,t} \to X_{st}$, thus there also has to be an irreducible map $\tau X_{st} \to X_{s+1,t}$. We know the Auslander-Reiten sequence ending in $X_{s+1,t}$; there are at most two arrows ending in $X_{s+1,t}$, one starting in $X_{s+1,t+1}$. There is a second such arrow in case $t \ge s+2$, and it starts in $X_{s+2,t}$. However, by induction we know that $\tau^- X_{s+2,t}$ is isomorphic to $X_{s+1,t-1}$, and the objects X_{st} and $X_{s+1,t-1}$ are not isomorphic. Thus the only possible choice for τX_{st} is $X_{s+1,t+1}$.

We consider the corresponding Auslander-Reiten sequence

$$0 \to X_{s+1,t+1} \xrightarrow{\begin{bmatrix} a \\ b \end{bmatrix}} Z \oplus X_{s+1,t} \xrightarrow{[-c\ d]} X_{st} \to 0,$$

and we can assume that $d = u_{s+1,t}$. Since this sequence is exact, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{s+1,t+1} & \xrightarrow{a} & Z & \xrightarrow{qc} & X_{ss} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \| & & \\
0 & \longrightarrow & X_{s+1,t} & \xrightarrow[u_{s+1,t}]{} & X_{st} & \xrightarrow[q]{} & X_{ss} & \longrightarrow & 0
\end{array}$$