æ

The Liu-Schulz Example

Claus Michael Ringel

ABSTRACT. Liu Shiping and Rainer Schulz have constructed a symmetric algebra $\Lambda{=}\Lambda(q)$ of dimension 8 and an indecomposable Λ-module M such that all the syzygy modules $\Omega^t M$ are 4-dimensional and pairwise non-isomorphic. In this way, they have exhibited an Auslander-Reiten component which contains infinitely many isomorphism classes of modules of dimension 4. The algebra $\Lambda(q)$ depends on a non-zero parameter $q{\in}k$, and the interesting behaviour occurs when q is not a root of unity. On the other hand, the case $q{=}1$ yields the exterior algebra $\Lambda(1)$ on the 3-dimensional vector space k^3, thus we may consider $\Lambda(q)$ as a sort of *quantization* of $\Lambda(1)$. The purpose of this note is to present a more detailed study of properties of this algebra $\Lambda{=}\Lambda(q)$ and of some of the Λ-modules.

Let k be a field, let R be a finite-dimensional k-algebra. We denote by τ (= $D\,\mathrm{Tr}$) the Auslander-Reiten translation in the category of finite-dimensional R-modules. Liu Shiping and Rainer Schulz have exhibited in [LS] an 8-dimensional algebra Λ and an indecomposable Λ-module M such that all the modules $\tau^t M$ (with $t \in \mathbb{Z}$) are 4-dimensional cyclic right ideals of Λ and pairwise non-isomorphic. In this way, they have answered a question raised in [R].

The algebra Λ is symmetric, thus $\tau = \Omega^2$, where Ω is Heller's loop functor, and even all the modules $\Omega^t M$ (with $t \in \mathbb{Z}$) are 4-dimensional cyclic right ideals and pairwise non-isomorphic. Such a behaviour would be impossible for some other classes of algebras such as group algebras of finite groups. We recall the following: let G be a finite group and kG its group algebra, let X be an indecomposable non-projective kG-module. Eisenbud [E] has shown: if the modules $\Omega^t X$ (with $t \in \mathbb{N}$) are of bounded dimension, then $\Omega^t X \simeq X$ for some $t \geq 1$ ("the modules of complexity 1 are just the periodic ones").

The algebra $\Lambda = \Lambda(q)$ constructed by Liu-Schulz depends on a non-zero parameter $q \in k$, and the interesting behaviour mentioned above occurs when q is not a root of unity. The purpose of this note is to present a detailed study of properties of the algebra $\Lambda(q)$ and of some of the $\Lambda(q)$-modules. The case $q = 1$ yields the

1991 *Mathematics Subject Classification.* Primary 16G70. Secondary 16S80, 16W50, 16L60.

exterior algebra $\Lambda(1)$ on the 3-dimensional vector space k^3, and we may consider $\Lambda(q)$ as a sort of "quantization" of $\Lambda(1)$. Also, over a field of characteristic 2, the algebra $\Lambda(1)$ is just the group algebra kE, where E is the elementary abelian group of order 8.

The example exhibited by Liu and Schulz is very interesting. The reader should observe that the features mentioned below should be considered not as pathological ones; in contrast, it seems that they are rather natural, and that it will be easy to construct many other algebras with similar properties. We may take these considerations as an indication that the representation theory of finite-dimensional algebras still is at its beginning: there may be many other basic features which still have to be discovered.

We are going to consider the class $\mathcal{M}$ of 4-dimensional cyclic right ideals of $\Lambda(q)$ in detail. We want to show the following: First, if M belongs to $\mathcal{M}$, then also the syzygy module ΩM belongs to $\mathcal{M}$. Second, the Auslander-Reiten component containing a module in $\mathcal{M}$ is of the form $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\tau^n$ for some n, and all the modules on the boundary of the component belong to $\mathcal{M}$.

The algebras $A(q)$ and $\Lambda(q)$ considered below are special cases of Sklyanin algebras and their Koszul duals, as studied by Artin, Tate, van den Bergh [ATB], by Paul Smith [Sm2,Sm3,Sm4] and others. We recommend the report [Sm4] by Smith which outlines the relevant concepts; the Liu-Schulz example is mentioned there in Remark 10.9. The present note may be considered as an elementary introduction. The author is endebted to Paul Smith for helpful comments.

1. The algebra $\Lambda = \Lambda(q)$

Let q be a non-zero element of k. We define the following algebra $\Lambda = \Lambda(q)$ using generators and relations.

As **generators** we take $x_0,\ x_1,\ x_2$; the **relations** are

$$x_i^2 = 0, \quad \text{and} \quad x_{i+1}x_i + q\cdot x_ix_{i+1} = 0 \quad \text{for} \quad i = 0, 1, 2$$

(here, and in the following, $x_3 = x_0$; in fact, we may take as indices residue classes of integers modulo 3).

Remark 1. Note that $\Lambda(q)$ may be considered as a q-deformation of $\Lambda(1)$ and $\Lambda(1)$ is just the exterior algebra on k^3 (another quantization of the exterior algebra $\Lambda(1)$ on k^3 has been considered in [Sm1]: there, one takes the relations $x_jx_i = q\cdot x_ix_j$ for $i < j$, and $x_i^2 = 0$). For k a field of characteristic 2, we may interprete $\Lambda(1)$ also as the group algebra of the elementary abelian group of order 8.

Remark 2. The considerations below will show that the ideal I generated by x_0 is 4-dimensional, and it is easy to see that Λ is isomorphic to the trivial extension of Λ/I by I. This is the way the algebra Λ is exhibited in the Liu-Schulz paper [LS]. Note however that in this presentation the 3-fold symmetry which will be of interest in all our considerations, remains hidden.