HALL ALGEBRA IN A ROOT CATEGORY

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Abstract In order to realize the quantum group in a global way, the Hall algebra of a root category (which is a orbit category of the derived category of a hereditary algebra) is constructed. The PBW-basis, filtration structure and integral form are investigated. Their degeneration at $v = 1$ actually provide a realization of the universal enveloping algebra of semisimple Lie algebras.

1. Introduction

1.1 Let $C = (a_{ij})_{ij}$ be the Cartan matrix of a finite dimensional semisimple complex Lie algebra $\mathbf{g}$ of rank n, this is a symmetrizable matrix, we denote by $(\varepsilon_i)_i$ the minimal symmetrization, thus ε_i are integers without a proper common divisor such that $\varepsilon_i a_{ij} = \varepsilon_j a_{ji}$. Take a variable v, $\mathbf{Q}(v)$ is the function field over the field $\mathbf{Q}$ of rational numbers. The quantum group $\mathcal{U} = \mathcal{U}_q(\mathbf{g})$ introduced by Drinfeld and Jimbo is an associative $\mathbf{Q}(v)$-algebra on the generators: $F_1, \cdots, F_n, K_1, \cdots, K_n, K_1^{-1}, \cdots, K_n^{-1}, E_1, \cdots, E_n$ subject to the relations:

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$K_iE_j = v_i^{a_{ij}}E_jK_i,\ K_iF_j = v_i^{-a_{ij}}F_jK_i$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$$

$$\sum_{t=0}^{1-a_{ij}}(-1)^t\left[\begin{array}{c}1-a_{ij}\\ t\end{array}\right]_{\varepsilon_i}E_i^tE_jE_i^{1-a_{ij}-t} = 0 \ \text{ for } \ i \neq j$$

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We admit the notations $q = v^2$, $v_i = v^{\varepsilon_i}$ and

$$[s] = \frac{v^s - v^{-s}}{v - v^{-1}} = v^{s-1} + v^{s-3} + \cdots + v^{-s+1}$$

$$[s]! = \prod_{r=1}^{s} [r], \quad \begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]!}{[r]![s-r]!}$$

and, for given a polynomial $f \in \mathbf{Z}[v, v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a. $\mathcal{U}$ has the triangular decomposition

$$\mathcal{U} \simeq \mathcal{U}^- \otimes \mathcal{T} \otimes \mathcal{U}^+$$

where $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) is the subalgebra generated by E_i (resp. F_i), $i = 1, \cdots, n$ and $\mathcal{T}$ generatd by $K_i^{\pm}, i = 1, \cdots, n$.

1.2 Now several programs for the realization of quantum group have been raised. A successful model is the Hall algebra of hereditary algebra which is introduced by Ringel [15].

Let k be a finite field, Λ be a finite dimensional hereditary k-algebra of finite representation type. For a Λ-module M, denoted by $\underline{\dim} M$ its dimension vector, it is an element in the Grothendieck group $G_0(\Lambda)$. It is well-known from the Gabriel theorem (see [4] and [3]) that Λ is characterized by the Dynkin quivers and $\underline{\dim}$ furnishes a bijection between the isomorphism classes of the indecomposable Λ-modules and the positive roots of the corresponding simple Lie algebra $\mathbf{g}$. It is natural to ask whether it is possible to recover a Lie algebra structure in terms of the representation theory of Λ. This was successfully attained by Ringel ([16], [17]). Not only this, he actually found a remarkable algebra structure from $\operatorname{mod} \Lambda$ which gives a realization of the positive part $\mathcal{U}^+$ of the quantum group $\mathcal{U}$. This is the Hall algebra (see [15], [18]).

By the Krull-Schmidt theorem and Gabriel theorem, we can identify the isomorphism classes of finite dimensional Λ-modules with the set $\{\varphi : \Phi^+ \to \mathbf{N_0}\}$. The bijection is given by

$$M(\varphi) = \bigoplus_{\alpha \in \Phi^+} \varphi(\alpha) M(\alpha)$$

where $M(\alpha)$ is indecomposable Λ-module with $\underline{\dim} M(\alpha) = \alpha$.

Let I be the index set of simple Λ-modules , $S_i = M(e_i)$ is the simple Λ-module corresponding to $i \in I$. For $\varphi, \psi : \Phi^+ \to \mathbf{N_0}$, we write

$$\langle \varphi, \psi \rangle = \langle M(\varphi), M(\psi) \rangle$$

$$= \dim_k \operatorname{Hom}(M(\varphi), M(\psi)) - \dim_k \operatorname{Ext}(M(\varphi), M(\psi))$$