DRINFELD DOUBLE AND GREEN-RINGEL THEORY OF HALL ALGEBRAS

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Abstract As a continuation to the work of Ringel and Green on Hall algebras, the Hopf algebra structure of a (extended twisted) Hall algebra is presented here. By introducing the Ringel pairing of Hall algebras, the Drinfeld double of a Hall algebra is constructed. The reduced form of this Drinfeld double is appliable to the generic composition algebra, which gives a complete realization of quantum group. This Green-Ringel theory make us possible give the formulae for the computation of commutators in quantum group in terms of Hall algebra.

1. Introduction

1.1 Let $C = (a_{ij})_{ij}$ be a symmetrizable generalized Cartan matrix, we denote by $(\epsilon_i)_i$ the minimal symmetrization, thus ϵ_i are integers without a proper common divisor such that $\epsilon_i a_{ij} = \epsilon_j a_{ji}$. Also, according to Lusztig [8], the so-called Cartan datum is an equivalent object with C (see (2.5)). Let $\mathbf{Q}(v)$ be the function field in one variable v over the field $\mathbf{Q}$ of rational numbers. The quantum group $\mathbf{U} = \mathbf{U}_q(\mathbf{g})$ introduced by Drinfeld and Jimbo is an associative $\mathbf{Q}(v)$-algebra on the generators: $F_1, \cdots, F_n, K_1, \cdots, K_n, K_1^{-1}, \cdots, K_n^{-1}, E_1, \cdots, E_n$ subject to the relations:

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$K_iE_j = v_i^{a_{ij}}E_jK_i,\ K_iF_j = v_i^{-a_{ij}}F_jK_i$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \left[\begin{array}{c} 1-a_{ij} \\ t \end{array} \right]_{\epsilon_i} E_i^t E_j E_i^{1-a_{ij}-t} = 0 \quad \text{for} \quad i \neq j$$

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We admit the notations $q = v^2$, $v_i = v^{\epsilon_i}$ and

$$[s] = \frac{v^s - v^{-s}}{v - v^{-1}} = v^{s-1} + v^{s-3} + \cdots + v^{-s+1}$$

$$[s]! = \prod_{r=1}^{s} [r], \quad \begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]!}{[r]![s-r]!}$$

and, for given a polynomial $f \in \mathbf{Z}[v, v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a. $\mathbf{U}$ has the triangular decomposition

$$\mathbf{U} \simeq \mathbf{U}^- \otimes \mathbf{T} \otimes \mathbf{U}^+$$

where $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) is the subalgebra generated by E_i (resp. F_i), $i = 1, \cdots, n$ and $\mathbf{T}$ generatd by $K_i^{\pm}, i = 1, \cdots, n$.

A strong feature of $\mathbf{U}$ is its Hopf algebra structure, however the Hopf operations is not closed in the positive part $\mathbf{U}^+$ (resp. $\mathbf{U}^-$), but it works for the Borel part $\mathbf{U}_q(\mathbf{b}^+)$ (resp. $\mathbf{U}_q(\mathbf{b}^-)$) of $\mathbf{U}$. Drinfeld found a general construction, the Drinfeld double, mirroring a Lie bialgebra. Applying the Drinfeld construction to the Borel parts $\mathbf{U}_q(\mathbf{b}^+)$ and $\mathbf{U}_q(\mathbf{b}^-)$, this gives $\mathbf{U} = \mathbf{U}_q(\mathbf{g})$ up to passage to a quotient (modulo a trivial Hopf ideal). We get this idea from [7].

1.2 Now several programs for the realization of quantum group have been raised. A successful model is the Hall algebra of hereditary algebra which is introduced by Ringel [13].

Let k be a finite field, Λ a finite dimensional hereditary k-algebra, $\mathcal{P}$ the set of isomorphism classes of finite dimensional Λ-modules. For every $\alpha \in \mathcal{P}$, choose u_α to be a representative in α, for $\alpha, \beta, \lambda \in \mathcal{P}$, let $g_{\alpha\beta}^{\lambda}$ be the number of submodules B of u_λ such that $B \simeq u_\beta$ and $u_\lambda / B \simeq u_\alpha$. The bilinear form $\langle -, - \rangle$ is given by $\langle \alpha, \beta \rangle = \dim_k \operatorname{Hom}_\Lambda(u_\alpha, u_\beta) - \dim_k \operatorname{Ext}_\Lambda(u_\alpha, u_\beta)$. The (twisted) Hall algebra $\mathbf{H}(\Lambda)$ is defined to be a $\mathbf{Q}$-space with the basis $\{u_\alpha \mid \alpha \in \mathcal{P}\}$, which multiplication is given by

$$u_\alpha u_\beta = \sum_{\lambda \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^{\lambda} u_\lambda$$

Ringel [13] has proved that the quantum serre relation is a zero element in Hall algebra and, if Λ and $\mathbf{g}$ enjoy a common Cartan datum of finite type, the generic form of $\mathbf{H}(\Lambda)$ gives a realization of $\mathbf{U}^+$ of $\mathbf{U}_q(\mathbf{g})$. Recently Green [5] found a remarkable formula about the filtration numbers of modules over a hereditary algebra, this led to a formulation for the comultiplication of $\mathbf{H}(\Lambda)$, he proved (depends on Lusztig [8]) that the positive part $\mathbf{U}^+$ of $\mathbf{U}_q(\mathbf{g})$ is canonically isomorphic to the generic composition algebra of Λ if $\mathbf{g}$ and Λ enjoy a common Cartan datum (of any type), where composition algebra is the subalgebra of the Hall algebra generated by simple Λ-modules.