TRIANGULAR DECOMPOSITION OF THE COMPOSITION ALGEBRA OF THE KRONECKER ALGEBRA

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Abstract. Let A be a finite dimensional hereditary algebra over a finite field, $\mathcal{H}(A)$ and $\mathcal{C}(A)$ be respectively the Ringel-Hall algebra and the composition algebra of A. Define $r_{\mathbf{d}}$ to be the element $\sum[M] \in \mathcal{H}(A)$, where $[M]$ runs over the isomorphism classes of the regular $A-$modules with dimension vector $\mathbf{d}$. We prove that $r_{\mathbf{d}}$ and the exceptional $A-$modules all lie in $\mathcal{C}(A)$. Let K be the Kronecker algebra, $\mathcal{P}$ (resp. $\mathcal{I}$) the subalgebra of $\mathcal{C}(K)$ generated by the preprojective (resp. preinjective) $K-$modules, and $\mathcal{T}$ the subalgebra generated by $r_{(n,n)}$ for $n \geq 0$, then we prove that $\mathcal{C}(K) = \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ and then $\mathcal{T}$ is just the subalgebra of $\mathcal{C}(K)$ generated by all regular elements.

0. Introduction and Preliminaries.

Throughout this paper A is assumed to be a finite dimensional hereditary k-algebra, where k is a finite field with q elements. All modules are finite dimensional over k (hence finite as sets). For the representation theory of finite dimensional hereditary algebras we refer to [R1].

By the recent works of C.M.Ringel [R3-9] and J.A.Green [G, Theorem 2] we know that Ringel's twisted generic composition algebra $\mathcal{C}_*(A)$ is isomorphic to G. Lusztig's algebra $\mathbf{f}$ (and hence isomorphic to the positive part U^+ of the Drinfeld-Jimbo quantization $U = U^- \otimes U^0 \otimes U^+$ associated to a Kac-Moody Lie algebra of type $(I, \cdot)$ ([L], chap.33), where $(I, \cdot)$ is the Cartan datum associated to A). This remarkable result provides the Ringel-Hall algebra approach to quantum group and makes it possible that results in the representations of quiver have applications or interpretations in quantum group.

To be simple, we consider the untwisted case (all considerations remain true under the twisted multiplication introduced in [R7]). Let $\mathcal{H}(A)$ and $\mathcal{C}(A)$ denote the Ringel-Hall algebra and Ringel's composition algebra of A respectively. By definition $\mathcal{H}(A)$ is the $\mathbb{Q}$-vector space with basis the set of isomorphism classes $[X]$ of finite modules, with the multiplication

$$[N_1] \cdot [N_2] := \sum_{[M]} F^M_{N_1,N_2}[M]$$

where $F^M_{N_1,N_2}$ is the number of the submodule N of M with $N \cong N_2$ and $M/N \cong N_1$. Then $\mathcal{H}(A)$ is an associative algebra with the identity $[0]$. And $\mathcal{C}(A)$ is by definition the subalgebra of $\mathcal{H}(A)$ generated by the elements $[S_i]$ for $1 \leq i \leq n$, where $S_i (1 \leq i \leq n)$ are all simple $A-$modules (up to isomorphism).

Given an isomorphism class $[M]$, we assign to its degree $\mathbf{dim} M \in \mathbb{Z}^n$, where $(\mathbf{dim} M)_i$ is the Jordan-Hölder multiplicity of S_i in M. Let $\mathcal{H}(A)_{\mathbf{d}}$ be the free abelian group with basis the set of isomorphism class $[M]$ with $\mathbf{dim} M = \mathbf{d}$, and $\mathcal{C}(A)_{\mathbf{d}} = \mathcal{H}(A)_{\mathbf{d}} \cap \mathcal{C}(A)$, then $\mathcal{H}(A) = \bigoplus\limits_{\mathbf{d}} \mathcal{H}(A)_{\mathbf{d}}$ and $\mathcal{C}(A) = \bigoplus\limits_{\mathbf{d}} \mathcal{C}(A)_{\mathbf{d}}$ are both $\mathbb{Z}^n$-graded ring. Let $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{Z}^n$ and $d := \sum\limits_i d_i$, then $\mathcal{C}(A)_{\mathbf{d}}$ is the $\mathbb{Q}$-subspace of $\mathcal{H}(A)$ spanned by all monomials $[S_{i_1}] \cdots [S_{i_d}]$, such that the number of occurrences of i in the sequence $i_1, \cdots, i_n$ is just d_i for $1 \leq i \leq n$. In this way we get all elements in $\mathcal{C}(A)$.

If A is representation-finite, Ringel has proved that $\mathcal{C}(A)$ coincides with $\mathcal{H}(A)$ ([R4], Prop.6); otherwise, $\mathcal{C}(A)$ is a proper subalgebra of $\mathcal{H}(A)$. In this case it is then of interest to ask which kinds of isomorphism

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classes $[M]$, or their combinations, lie in the composition algebra: one of the advantages of the Ringel-Hall algebra approach to quantum group is that it has a natural basis. We present two results towards this direction in Section 1, see Theorem 1 and 2 below.

Since the preprojective and preinjective modules lie in $\mathcal{C}(A)$, it is natural to consider the regular modules. In Section 2 we use the notation of the regular part to obtain more regular elements in $\mathcal{C}(K_m)$ (Theorem 3), where K_m is the generalized Kronecker algebra.

Finally we apply these results to determine the structure of $\mathcal{C}(K)$ (Theorem 4), where K is the Kronecker algebra. It turns out that $\mathcal{C}(K)$ shares a triangular decomposition (this may be an analogy to the one of the Kac-Moody Lie algebra $A_2^{(1)}$ and the quantum group $U(A_2^{(1)})$). From this a PBW-basis of $\mathcal{C}(K)$ can be explicitly written out.

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1. Exceptional modules.

An indecomposable $A-$module M is said to be preprojective (resp. preinjective; regular) provided that $\tau^m M = 0$ for some $m > 0$ (resp. $\tau^m M = 0$ for some $m < 0$; $\tau^m M \neq 0$ for all $m \in \mathbb{Z}$), where τ is the Auslander-Reiten translate. An $A-$module is said to be preprojective (resp. preinjective; regular) provided that each indecomposable direct summand is so.

For $\mathbf{d} \in \mathbb{N}_0^n$, denote by $r_{\mathbf{d}}$ the element $\sum\limits_{[M]}[M] \in \mathcal{H}(A)$, where $[M]$ runs over the isomorphism classes of the regular modules with $\mathbf{dim}M = \mathbf{d}$ (since A is finitary (see [R2]), this sum is of course a finite sum). Note that M may be decomposable; and if there exists no regular module M with $\mathbf{dim}M = \mathbf{d}$, then $r_{\mathbf{d}} := [0]$.

Theorem 1. (i) The preprojective and preinjective $A-$modules lie in the composition algebra $\mathcal{C}(A)$.

(ii) For $\mathbf{d} \in \mathbb{N}_0^n$, $r_{\mathbf{d}} \in \mathcal{C}(A)$.

In order to prove Theorem 1 we need the following two lemmas, which will also be used later.

Lemma 1. Let $\mathbf{d} \in \mathbb{N}_0^n$.

(i) If $\mathbf{d} = \mathbf{dim}P$ with P an indecomposable preprojective A-modules, then $r_{\mathbf{d}} = [0]$, i.e. there is no regular module M with $\mathbf{dim}M = \mathbf{d}$.

(ii) If $\mathbf{d} = \mathbf{dim}I$ with I an indecomposable preinjective A-modules, then $r_{\mathbf{d}} = [0]$.

Proof. (i) Let $< -,- >$ be the Ringel form, i.e. $< \mathbf{dim}X, \mathbf{dim}Y >= dim_k Hom_A(X,Y) - dim_k Ext^1_A(X,Y)$ for $X, Y \in A-mod$. If there exists a regular module M with $\mathbf{dim}M = \mathbf{d}$, then we get a contradiction:

$$\begin{aligned} 1 &\leq < \mathbf{dim}P, \mathbf{dim}P > = < \mathbf{dim}M, \mathbf{dim}P > \\ &= dim_k Hom(M,P) - dim_k Ext^1(M,P) \\ &= -dim_k Ext^1(M,P). \end{aligned}$$

(ii) The dual of (i). □

Let $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{N}_0^n$, we define $\mathbf{x} \prec \mathbf{y}$ if and only if $x_i \leq y_i$, $\sum\limits_{i=1}^n x_i < \sum\limits_{i=1}^n y_i$. This gives a partial order in $\mathbb{N}_0^n$. For $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{N}_0^n$, denote by $\prec_{\mathbf{d}}$ the subset of $\mathbb{N}_0^n$ of elements $\mathbf{x} \prec \mathbf{d}$.