ON THE NORM RESIDUE HOMOMORPHISM FOR FIELDS

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Abstract. It is proved that the triviality of the Bockstein map of degree n for all fields of characteristic different from 2 implies the bijectivity of the norm residue homomorphism of degree n modulo 2. In particular, the bijectivity of the norm residue homomorphism follows from its surjectivity.

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Let F be a field and m be a natural number prime to the characteristic of F. The interaction between Milnor K-theory and Galois cohomology of the field F is presented by the *norm residue homomorphism* of degree n modulo m:

$$h_{n,m,F} : K_n F/mK_n F \to H^n(F, \mu_m^{\otimes n}).$$

There is a well-known conjecture saying that $h_{n,m,F}$ is always an isomorphism. This is clear if $n = 0$, simply follows from Hilbert theorem 90 for $n = 1$, and is known to be true if $n = 2$ for arbitrary m [8]. It is also known to be true if m is a power of 2 and $n = 3$ [9],[14] or $n = 4$ (Rost, unpublished) for an arbitrary field F.

In the present paper we consider the case of arbitrary n, $m = 2$ and a field F of characteristic different from 2. It is easy to see (Lemma 2.1) that the surjectivity of the norm residue homomorphism $h_n = h_{n,2,F}$ implies the triviality of the Bockstein connecting map of degree n

$$B_n = B_{n,F} : H^n(F, \mu_2^{\otimes n}) \to H^{n+1}(F, \mu_2^{\otimes n})$$

given by the short exact sequence

$$1 \to \mu_2^{\otimes n} \to \mu_4^{\otimes n} \to \mu_2^{\otimes n} \to 1.$$

We prove that, conversely, the purely cohomological property of the Bockstein map to be trivial implies the bijectivity of the norm residue homomorphism h_n. More precisely, our main result is the following

Theorem. *Suppose that for some $n \geqslant 0$ the Bockstein map B_n is trivial for all fields. Then the norm residue homomorphism h_n is an isomorphism for all fields.*

Corollary. *The surjectivity of h_n for all fields implies the bijectivity of h_n also for all fields.*

1. Notation and preliminary results

Milnor K-theory. Let F be a field. The Milnor graded ring K_*F of the field F is the factor ring of the tensor $\mathbb{Z}$-algebra $T(F^*)$ modulo the ideal generated by the tensors $a\otimes(1-a)$ for all $a\neq 0,1$ in F [13]. Hence, $K_*F=\coprod_{n\geqslant 0}K_nF$, $K_0F=\mathbb{Z}$, $K_1F=F^*$.

For any field extension E/F there is a natural ring homomorphism $i_{E/F}\colon K_*F\to K_*E$ induced by the inclusion $F\hookrightarrow E$.

If E/F is a finite extension, then there is defined the norm homomorphism $n_{E/F}: K_*E\to K_*F$ [5]. The *projection formula* $n_{L/F}(u\cdot i_{L/F}(v))=n_{L/F}(u)\cdot v$ holds for any $u\in K_*E$, $v\in K_*F$. In particular the composition $n_{E/F}\circ i_{E/F}$ is the multiplication by the degree $[E:F]$ of the extension E/F.

Throughout the paper we use the notation k_*F for $K_*F/2K_*F$. The image of the tensor $a_1\otimes a_2\otimes\ldots\otimes a_n$ in k_*F will be denoted by $\{a_1,a_2,\ldots,a_n\}$ and called a *symbol.*

If E/F is a quadratic extension, then k_nE is generated by symbols of the type $u=\{x,a_1,a_2,\ldots a_{n-1}\}$ where $x\in E^*$, $a_i\in F^*$ [2] and

$$n_{E/F}(u)=\{n_{E/F}(x),a_1,a_2,\ldots a_{n-1}\}$$

by the projection formula.

Let X be an algebraic variety defined over F. For any $i\geqslant 0$ denote by $X^{(i)}$ the subset of points in X of codimension i. The group

$$C_n^i(X)=\coprod_{x\in X^{(i)}}k_nF(x)$$

is called the group of k_n*-cycles of codimension* i *on* X. There is the complex [6]:

$$\cdots\to C_{n+1}^{i-1}(X)\to C_n^i(X)\to C_{n-1}^{i+1}(X)\to\ldots.$$

We denote by $H^i(X,k_{n+i})$ the homology group of this complex in $C_n^i(X)$.

If $X=\mathbb{A}_F^1$ is the affine line, then $H^0(X,k_n)=k_nF$ and $H^i(X,k_n)=0$ if $i>0$. In other words, there is an exact sequence [13]:

$$0\to k_nF\to k_nF(t)\to\coprod_{x\in\mathbb{A}^1}k_{n-1}F(x)\to 0.$$

Cohomology. Denote by F_s a separable closure of the field F, and let $\Gamma=\mathrm{Gal}(F_s/F)$. For any discrete Γ-module M by $H^*(F,M)$ we denote the cohomology groups $H^*(\Gamma,M)$ [16].

For any field extension E/F there is a natural *restriction* homomorphism:

$$i_{E/F}\colon H^*(F,M)\to H^*(E,M).$$

If E/F is a finite extension, then there exists the *norm homomorphism* (*corestriction*):

$$n_{E/F}\colon H^*(E,M)\to H^*(F,M).$$