QUASI POLYMATROIDAL FLOW NETWORKS

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Abstract. In this paper we give a flow model on directed multigraphs by introducing reflexions of generalized polymatroids at vertices as constraints for the flow conservation. This model has the essential features of the classical flow model, primarily the max-flow min-cut theorem and the polynomial algorithm for computing the maximal feasible (integral) flow.

1. Introduction

Since the classical network flow model of Ford and Fulkerson appeared in the 1950's, numerous generalizations and variations of this model have been introduced. Very interesting is the polymatroidal network flow model introduced by Lawler and Martel [12] and Hassin [9] which provides a generalization and unification of both network flow theory and much of the theory of polymatroid optimization (see [13]).

Note that (integral) polymatroids are polyhedra of nonnegative vectors bounded by (integral) submodular function. They have been introduced by Edmonds [2] as generalizations of matroids. (Integral) polymatroids and the polyhedra arising as intersections of two (integral) polymatroids play a very important role in (integral) optimization. Further generalizations of polymatroids are generalized polymatroids, that are polyhedra bounded by sub- and supermodular functions having an additional property. They were introduced by Frank [4]. A comprehensive survey of their properties can be found in Frank and Tardos [6].

A *polymatroidal flow network* $\mathcal{F}$ (see Lawler and Martel [12]) is a directed multigraph G with a source s, a sink t and for any vertex v of G we have two polymatroids $\mathbb{P}_v^+$, $\mathbb{P}_v^-$ on the set of arcs entering (leaving) v respectively. A flow f is said to be *feasible* in $\mathcal{F}$ if for any vertex v of G, the vector whose coordinates are the values of f on the arcs entering (leaving) v is independent in $\mathbb{P}_v^+$ ($\mathbb{P}_v^-$), (i. e., the vector is an element of the polytope of this polymatroid). Furthermore, $\mathcal{F}$ is called *integral* if $\mathbb{P}_v^+$, $\mathbb{P}_v^-$ are integral for any vertex v of G.

In [12] it was proved that this model has the max-flow min-cut property and a polynomial algorithm that computes a maximal (integral) flow in an (integral) polymatroidal flow network was introduced. Hassin [9] dealt with a circulant variant of this model. Another generalization of this model was introduced by Lawler and Martel [14] (see also Tardos, Tovey and Trick [22]). As pointed out in [13],

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polymatroidal network flow model is a very useful theoretical and practical tool in combinatorial optimization and many polymatroid optimization problems can be easily formulated and solved in the terms of this flow model. Other models of combinatorial optimization equivalent with the polymatroidal network flow model are surveyed in Schrijver [20].

An undirected analogue of the polymatroidal network flow model is in fact the Matchoid problem (see Lovász and Plummer [16]). This model has its fount in matching theory and was originated by Edmonds (see [16] for more details).

The matchoid problem: Let G be an undirected graph such that for any vertex v of G a matroid M_v is given on the set of the edges adjacent with v. Call a set M of edges a matchoid if, for any vertex v of G, the set of the edges of M adjacent with v is independent in M_v. Find a matchoid with the maximum cardinality.

Lovász [15] proved that the matchoid problem is NP-hard and that every oracle algorithm for this problem has exponential complexity. This is in contrast with algorithms for computing the integral polymatroidal flow networks.

Let us stress the fact that in the polymatroidal network flow model there are no correlations between the constraints imposed on the set of arcs leaving and the set of arcs entering a vertex (in other words, $\mathbb{P}_v^+$ and $\mathbb{P}_v^-$ are independent of each other for any vertex v). Any feasible flow must satisfy only a natural condition that, for any vertex v different from the source and the sink, the sum of the values of f on the arcs entering v is equal to the sum of the values of f on the arcs leaving v. Flow models with this property we shall call *balanced*.

It is natural to ask the following question in connection with the two above models: Does there exist a reasonable flow model on digraphs (i. e., a model with polynomial algorithm for computing maximal feasible "integral" flow) in which the whole neighbourhood of each vertex (no matter on orientation) will be constrained by a single polymatroid or another relative polyhedra?

In this paper we answer this question affirmatively and introduce a network flow model on digraphs such that a flow f is feasible if for any vertex v the vector f_v is independent in a given generalized polymatroid, where f_v is the direct sum of two vectors f_v^+ and $-f_v^-$ such that the coordinates of f_v^+ (f_v^-) are the values of the flow f on the arcs entering (leaving) v. This model will be called the quasi polymatroidal network flow model. We show that it also has the essential features of the classical flow model (the max-flow min-cut theorem and polynomial algorithm for computing the maximal feasible "integral" flow).

Our flow model has a common feature with the well-known concept of "nowhere-zero flows" on graphs (see, e.g., Jaeger [10] for a survey of this topic). It consists in the fact, that if we interchange the orientation of an arc in a quasi polymatroidal flow network, we get a new flow network of the same type and, in a certain sense, equivalent with the original flow network. A more detailed discussion about this matter will be given in the fourth section.

As pointed out earlier the polymatroidal network flow model of Lawler and Martel is balanced. Our flow model is not balanced in general. This is also of some interest with respect to the fact that it is known that there exist unbalanced flow models on digraphs for which it is NP-hard to compute the maximal integral flow (see, e. g., Sahni [19] for the integral network flows with multipliers).

Note that our flow model is equivalent with the flow models of Lawler and Martel